Refinements of Milnor's Fibration Theorem for Complex Singularities☆

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Abstract

Let X be an analytic subset of an open neighbourhood U of the origin $\underline{0}$ in $\mathbb{C}^n$. Let $f\colon (X,\underline{0}) \to (\mathbb{C},0)$ be holomorphic and set $V = f^{-1}(0)$. Let $\mathbb{B}_\varepsilon$ be a ball in U of sufficiently small radius $\varepsilon > 0$, centred at $\underline{0} \in \mathbb{C}^n$. We show that f has an associated canonical pencil of real analytic hypersurfaces X_θ, with axis V, which leads to a fibration Φ of the whole space $(X \cap \mathbb{B}_\varepsilon) \setminus V$ over $\mathbb{S}^1$. Its restriction to $(X \cap \mathbb{S}_\varepsilon) \setminus V$ is the usual Milnor fibration $\phi = \frac{f}{|f|}$, while its restriction to the Milnor tube $f^{-1}(\partial \mathbb{D}_\eta) \cap \mathbb{B}_\varepsilon$ is the Milnor-Lê fibration of f. Each element of the pencil X_θ meets transversally the boundary sphere $\mathbb{S}_\varepsilon = \partial \mathbb{B}_\varepsilon$, and the intersection is the union of the link of f and two homeomorphic fibers of ϕ over antipodal points in the circle. Furthermore, the space $\tilde{X}$ obtained by the real blow up of the ideal $(Re(f), Im(f))$ is a fibre bundle over $\mathbb{RP}^1$ with the X_θ as fibres. These constructions work also, to some extent, for real analytic map-germs, and give us a clear picture of the differences, concerning Milnor fibrations, between real and complex analytic singularities.

Key words: Real and complex singularities, Milnor fibration, rugose vector fields, stratifications, pencils, open-books.
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Introduction

Milnor's fibration theorem is a key-stone in singularity theory. This is a result about the topology of the fibres of analytic functions near their critical points.

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Let X be an analytic subset of an open neighbourhood U of the origin $\underline{0}$ in $\mathbb{C}^n$. Given $f\colon (X, \underline{0}) \to (\mathbb{C}, 0)$ holomorphic with a critical point at $\underline{0} \in X$ (in the stratified sense), there are two equivalent ways of defining its *Milnor fibration*.

The first was given in [22, Thm. 4.8] for X smooth and, as we show below, extends for arbitrary X: let $\mathbb{B}_\varepsilon$ be a closed ball of sufficiently small radius ε around $\underline{0} \in \mathbb{C}^n$ and let $\mathbb{S}_\varepsilon$ be its boundary sphere, let $L_X = X \cap \mathbb{S}_\varepsilon$ be the link of X and let $L_f = f^{-1}(0) \cap \mathbb{S}_\varepsilon$ be the link of f in X. Then the fibration is:

$$\phi = \frac{f}{|f|}\colon L_X \setminus L_f \longrightarrow \mathbb{S}^1. \tag{1}$$

This fibration theorem, for general X, is implicit in the work of Lê Dũng Tráng [16] and a weaker form of it is also given in [8, Thm. 3.9].

The second version of the fibration theorem also originates in Milnor's book [22]. For this, choose $\varepsilon >> \eta > 0$ sufficiently small and consider the *Milnor tube*

$$N(\varepsilon, \eta) = X \cap \mathbb{B}_\varepsilon \cap f^{-1}(\partial \mathbb{D}_\eta)\,,$$

where $\mathbb{D}_\eta \subset \mathbb{C}$ is the disc of radius η around $0 \in \mathbb{C}$. Then

$$f\colon N(\varepsilon, \eta) \longrightarrow \partial \mathbb{D}_\eta\,, \tag{2}$$

is a fibre bundle, isomorphic to the previous bundle (1).[2] We notice that in his book, J. Milnor proves only that for X smooth, the fibres of (2) are isomorphic to those of (1), but he does not prove that (2) is actually a fibre bundle, a fact he certainly knew when f has an isolated critical point and $X = \mathbb{C}^n$ (see [21]). H. Hamm in [11] extended Milnor's work to the case when $X \setminus f^{-1}(0)$ is non singular, and D.T. Lê [15, Thm. (1.1)] proved that (2) is a fibre bundle in full generality. We call (2) the Milnor-Lê fibration, to distinguish it from the equivalent fibration (1).

In this work we improve, or refine, these fibration theorems in five directions, given by Theorems 1, 2, 3, 5 and 6.

The starting point, that originates in [31, 26], is to notice that every holomorphic map f as above determines a canonical pencil of real analytic hypersurfaces, with axis $V = f^{-1}(0)$, and this pencil gives rise to both fibrations (1) and (2) as we explain below.

For simplicity, denote also by X the intersection $X \cap \mathbb{B}_\varepsilon$. (See Section 1.2 for the definition of Whitney-strong stratifications.)

Theorem 1 (Canonical Decomposition). *For each $\theta \in [0, \pi)$, let $\mathcal{L}_\theta$ be the line through 0 in $\mathbb{R}^2$ with an angle θ (with respect to the x-axis). Set $V = f^{-1}(0)$ and $X_\theta = f^{-1}(\mathcal{L}_\theta)$. Then one has:*

[2] Throughout this article we speak of "equivalence" of these (and similar) fibrations. This statement must be made precise, since $N(\varepsilon, \eta)$ is compact and $L_X \setminus L_f$ is not; the second fibration must be restricted to L_X minus an open regular neighbourhood of L_f to have an actual equivalence, but this determines the whole fibration on $L_X \setminus L_f$ (see the proof of Theorem 2 in Section 3.1 below). The "equivalence" must be understood in this sense.

i) *The X_θ are all homeomorphic real analytic hypersurfaces of X with singular set $\mathrm{Sing}(V) \cup (X_\theta \cap \mathrm{Sing}(X))$. Their union is the whole space X and they all meet at V, which divides each X_θ in two homeomorphic halves.*

ii) *If $\{S_\alpha\}$ is a Whitney stratification of X adapted to V, then the intersection of the strata with each X_θ determines a Whitney-strong stratification of X_θ, and for each stratum S_α and each X_θ, the intersection $S_\alpha \cap X_\theta$ meets transversally every sphere in $\mathbb{B}_\varepsilon$ centred at $\underline{0}$.*

iii) *There is a uniform conical structure for all X_θ, i.e., there is a (rugose) homeomorphism*

$$h\colon (X \cap \mathbb{B}_\varepsilon, V \cap \mathbb{B}_\varepsilon) \to \big(Cone(L_X), Cone(L_f)\big),$$

which restricted to each X_θ defines a homeomorphism

$$(X_\theta \cap \mathbb{B}_\varepsilon) \cong Cone(X_\theta \cap \mathbb{S}_\varepsilon).$$

Theorem 2 (Fibration Theorem). *One has a commutative diagram of fibre bundles*

$$\begin{tikzcd} (X \cap \mathbb{B}_\varepsilon) \setminus V \arrow[r, "\Phi"] \arrow[dr, "\Psi"'] & \mathbb{S}^1 \arrow[d, "\pi"] \\ & \mathbb{RP}^1 \end{tikzcd}$$

where $\Psi(x) = (Re(f(x)) : Im(f(x)))$ with fibre $(X_\theta \cap \mathbb{B}_\varepsilon) \setminus V$, $\Phi(x) = \frac{f(x)}{\|f(x)\|}$ and π is the natural two-fold covering. The restriction of Φ to the link $L_X \setminus L_f$ is the usual Milnor fibration ϕ in (1), while the restriction of Φ to the Milnor tube $f^{-1}(\partial \mathbb{D}_\eta) \cap \mathbb{B}_\varepsilon$ is the Milnor-Lê fibration (2) (up to multiplication by a constant), and these two fibrations are equivalent.

To prove Theorem 2 we introduce in Section 2.2 the *spherefication* of f, which is an auxiliary function defined by $\mathfrak{F}(x) = \|x\|\Phi(x) = \|x\| \frac{f(x)}{\|f(x)\|}$. This map has the property that its "Milnor tubes" are precisely the "Milnor fibrations on the spheres". More precisely we have the following Fibration Theorem.

Theorem 3. *For $\varepsilon > 0$ sufficiently small, one has a fibre bundle*

$$\mathfrak{F}\colon \big((X \cap \mathbb{B}_\varepsilon) \setminus V\big) \longrightarrow (\mathbb{D}_\varepsilon \setminus \{0\}),$$

taking x into $\|x\| \frac{f(x)}{\|f(x)\|}$, where $\mathbb{D}_\varepsilon$ is the disc in $\mathbb{R}^2$ centred at 0 with radius ε. Furthermore, the restriction of $\mathfrak{F}$ to each sphere around $\underline{0}$ of radius $\varepsilon' \leq \varepsilon$ is a fibre bundle over the corresponding circle of radius ε', and this is the Milnor fibration ϕ in (1) up to multiplication by a constant.

Our proofs actually show:

Corollary 4. *Let* $f\colon (X,\underline{0}) \to (\mathbb{C},0)$ *be as above, a holomorphic map with a critical point at* $\underline{0} \in X$*, and consider its Milnor fibration*

$$\phi = \frac{f}{|f|}\colon L_X \setminus L_f \longrightarrow \mathbb{S}^1 \,.$$

If the germ $(X,\underline{0})$ *is irreducible, then we have that every pair of fibres of* ϕ *over antipodal points of* $\mathbb{S}^1$ *are glued together along the link* L_f *producing the link of a real analytic hypersurface* X_θ*, which is homeomorphic to the link of* $\{Re\, f = 0\}$*. Moreover, if both* X *and* f *have an isolated singularity at* $\underline{0}$*, then this homeomorphism is in fact a diffeomorphism and the link of each* X_θ *is diffeomorphic to the double of the Milnor fibre of* f *regarded as a smooth manifold with boundary* L_f*.*

The above hypothesis of X being irreducible can be relaxed, assuming only that X and V do not share an irreducible component at the origin (see Remark 4.7).

Notice that for $\theta = \pi/2$ the corresponding variety X_θ is $\{Re\, f = 0\}$. Thus for instance, for the map $(z_1, z_2) \overset{f}{\mapsto} z_1^2 + z_2^q$ one gets that the link of $Re\, f$ is a closed, oriented surface in the 3-sphere, union of the Milnor fibres over the points $\pm i$; an easy computation shows that it has genus $q-1$. It would be interesting to study geometric and topological properties of the 4-manifolds one gets in this way, by considering the link of the hypersurface in $\mathbb{C}^3$ defined by the real part of a holomorphic function with an isolated critical point. For example, for the map $(z_1, z_2, z_3) \overset{f}{\mapsto} z_1^2 + z_2^3 + z_3^5$, the corresponding 4-manifold is the double of the famous E_8 manifold with boundary Poincare's homology 3-sphere.

In order to complete the proof of Theorem 1 we actually show (Section 3):

Theorem 5. *Let* $\tilde{X}$ *be the space obtained by the real blow-up of* V*, i.e., the blow-up of* $(Re(f), Im(f))$*. The projection* $\tilde{\Psi} : \tilde{X} \to \mathbb{RP}^1$ *is a topological fibre bundle with fibre* X_θ*.*

This result strengthens Theorem 2 and implies that all the hypersurfaces X_θ are homeomorphic: Are they actually analytically equivalent? we do not know the answer. The proof of Theorem 5 (in Section 4) can be refined to prove also Theorem 2. However we prefer to give (in Section 3) a direct proof of Theorem 2, which is elementary and throws light into the understanding of Milnor-type fibrations for real analytic maps, that we envisage in Section 5: We consider real analytic map-germs from $\mathbb{R}^{n+2}$ into $\mathbb{R}^2$. One has a canonical pencil (X_θ) associated to f as in the holomorphic case, but now the pencil may not have the uniform conical structure of Theorem 1. If it does, at least away from V, then we say that f is d-regular; we give various examples of families of such singularities. We prove:

Theorem 6. *Let* $f : (U,\underline{0}) \to (\mathbb{R}^2, 0)$ *be a locally surjective real analytic map with an isolated critical value at* $0 \in \mathbb{R}^2$*,* U *an open neighbourhood of* $\underline{0}$ *in* $\mathbb{R}^{n+2}$ *and* $V = f^{-1}(0)$*. Assume further that at* $\underline{0}$*,* f *has the Thom* a_f *property, it is d-regular and* $dim V > 0$*. Then:*

i) *One has a Milnor-Lê fibration (a fibre bundle)*

$$f\colon N(\varepsilon,\eta) \longrightarrow \partial\mathbb{D}_\eta\,,$$

where $N(\varepsilon,\eta) = \mathbb{B}_\varepsilon \cap f^{-1}(\partial\mathbb{D}_\eta)\,,$ *is a Milnor tube for* f*;* $\mathbb{D}_\eta \subset \mathbb{R}^2$ *is the disc of radius* η *around* $0 \in \mathbb{R}^2$, $\varepsilon >> \eta > 0$*. In fact this same statement holds for* $\mathbb{B}_\varepsilon \cap f^{-1}(\mathbb{D}_\eta \setminus 0)\,,$ *which fibres over* $\mathbb{D}_\eta \setminus 0$*.*

ii) *For every sufficiently small* $\varepsilon > 0$ *one has a commutative diagram of fibre bundles,*

$$\begin{tikzcd} \mathbb{B}_\varepsilon \setminus V \arrow[r, "\Phi"] \arrow[dr, "\Psi"'] & \mathbb{S}^1 \arrow[d, "\pi"] \\ & \mathbb{RP}^1 \end{tikzcd}$$

which induces by restriction, a fibre bundle $\mathbb{S}_\varepsilon \setminus K_\varepsilon \xrightarrow{\phi} \mathbb{S}^1$ *with* $\phi = f/|f|$ *and* $K_\varepsilon = V \cap \mathbb{S}_\varepsilon$*.*

iii) *The two fibrations above, one on the Milnor tube, the other on the sphere, are equivalent.*

We remark that in the holomorphic case, critical values must be isolated. This is not the case in general for real analytic maps into $\mathbb{R}^2$ (cf. for instance [25]). We remark also that if in Theorem 6 we remove the hypothesis that the image of f is a neighbourhood of $0 \in \mathbb{R}^2$, then the results are true restricting the fibrations to their images. This is carefully discussed in [6] where an example is given in [6, Remark 4.1].

The study of Milnor-type fibrations for real analytic mappings is a subject that goes back to Milnor's work in [21, 22], and there are several recent papers on the topic. We refer to [32, 33] for overviews on this, and to [5, 19, 23, 25, 26, 28, 29] for more recent work. Notice also that part of the content of Section 5 generalises to real analytic map-germs $(X, \underline{0}) \to (\mathbb{R}^k, 0)$ with $k \geq 2$ and X singular.

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1. Preliminaries

This section contains well-know material about stratified analytic spaces.

1.1. Stratifications and Whitney conditions

Let $\mathbb{F}$ be either the real or the complex numbers. A *stratification* of a subset X of $\mathbb{F}^n$ is a *locally finite* partition $\{S_\alpha\}$ of X into smooth, connected submanifolds of $\mathbb{F}^n$ (called *strata*) which satisfy the *Frontier Condition*, that if S_α and S_β are strata with $S_\alpha \cap \bar{S}_\beta \neq \emptyset$, then $S_\alpha \subset \bar{S}_\beta$.

When X is complex analytic, we say that a stratification $\{S_\alpha\}$ is *complex analytic* if all the strata are analytic. In the real analytic case, we say that $\{S_\alpha\}$ is *subanalytic* if each S_α, its closure $\bar{S}_\alpha$ and $\bar{S}_\alpha \setminus S_\alpha$ are subanalytic.

Now consider a triple (y, S_α, S_β), where S_α and S_β are strata of X with $y \in S_\alpha \subset \bar{S}_\beta$. We say that the triple (y, S_α, S_β) is *Whitney regular* if it satisfies the *Whitney* (b) *condition*:

i) given a sequence $\{x_n\} \subset S_\beta$ converging in $\mathbb{F}^n$ to $y \in S_\alpha$ such that the sequence of tangent spaces $T_{x_n}S_\beta$ converges to a subspace $T \subset \mathbb{F}^n$; and
ii) a sequence $\{y_n\} \subset S_\alpha$ converging to $y \in S_\alpha$ such that the sequence of lines (secants) $l_{x_i y_i}$ from x_i to y_i converges to a line l;

then one has $l \subset T$.

By convergence of tangent spaces or secants we mean convergence of the translates to the origin of these spaces, so these are points in the corresponding Grassmannian.

There is also a *Whitney* (a) *condition*: in the above situation i) one has that T contains the space tangent to S_α at y. It is an exercise to show that condition (b) implies condition (a).

Definition 1.1. The stratification $\{S_\alpha\}$ of X is *Whitney regular* (also called a *Whitney stratification*) if every triple (y, S_α, S_β) as above, is Whitney regular.

The existence of Whitney stratifications for every analytic space X was proved by Whitney in [37, Thm. 19.2] for complex varieties, and by Hironaka [12] in the general setting.

1.2. Whitney-strong stratifications

We now describe another regularity condition, defined by Verdier in [36], improving Kuo's regularity condition in [14]. For this let A and B be vector subspaces of $\mathbb{F}^n$, and let π_B be the orthogonal projection onto B. We define the *distance* (or *angle*) between A and B by

$$\delta(A, B) = \sup_{\substack{a \in A, \\ \|a\|=1}} \operatorname{dist}(a, B) = \sup_{\substack{a \in A, \\ \|a\|=1}} \|a - \pi_B(a)\|. \tag{3}$$

Notice that this is not symmetric in A and B. Also, $\delta(A, B) = 0$ if and only if $A \subseteq B$, and $\delta(A, B) = 1$ if and only if there exists $a \in A$ such that $a \perp B$. If B is a subspace of C, then $\delta(A, C) \leq \delta(A, B)$.

The *Kuo-Verdier* (w) *condition (also known as Whitney-strong condition)* for a triple (y, S_α, S_β) as above is that there exists a neighbourhood $\mathcal{U}_y$ of $y \in S_\alpha$ in $\mathbb{F}^n$ and a constant $C > 0$ such that

$$\delta(T_{y'}S_\alpha, T_x S_\beta) \le C\,\|y' - x\|$$

for all $y' \in \mathcal{U}_y \cap S_\alpha$ and all $x \in \mathcal{U}_y \cap S_\beta$.

Remark 1.2. Condition (w) reinforces condition (a), and for analytic stratifications condition (w) implies condition (b) ([36, Thm. (1.5)] or [14]). Moreover, for complex analytic stratifications, Teissier proved in [34] that conditions (b) and (w) are actually equivalent.

Definition 1.3. The stratification $\{S_\alpha\}$ is said to be *Whitney-strong* if every triple (y, S_α, S_β) satisfies condition (w).

The existence of Whitney-strong stratifications for all analytic spaces was proved by Verdier in [36, Thm. (2.2)], using Hironaka's theorem of resolution of singularities. There are simpler proofs in [18, 7].

The following theorem summarises results from [37] and [36].

Theorem 1.4. *Let $(X, \underline{0})$ be a (real or complex) analytic germ in $\mathbb{F}^n$, and let V be an analytic variety in X. Then we can endow X with a locally finite Whitney-strong analytic stratification such that V is union of strata, $X \setminus (V \cup \mathrm{Sing}(X)$ is a stratum (possibly disconnected in the real case). We can further assume $\underline{0}$ is a stratum and there exists a sufficiently small ball $\mathbb{B}$ around $\underline{0} \in \mathbb{F}^n$, such that each stratum contains $\underline{0}$ in its closure and is transverse to all the spheres in $\mathbb{B}$ centred at $\underline{0}$.*

This yields (compare with [22, Thm. 10.2] and [4, Lemma 3.2]):

Theorem 1.5. *For $\mathbb{B}$ as above, one has that the triple $(\mathbb{B}, X \cap \mathbb{B}, V \cap \mathbb{B})$ is homeomorphic to the cone over the triple $(\partial\mathbb{B}, X \cap \partial\mathbb{B}, V \cap \partial\mathbb{B})$.*

Definition 1.6. A Whitney-strong stratification of X as in Theorem 1.4 will be called a *Whitney stratification adapted to V*.

1.3. The Thom Property

We now look at regularity conditions for the space X relative to a function on it. This originates in the work of R. Thom [35].

Let $X \subset U \subset \mathbb{C}^n$ be a complex analytic subspace of an open set U of $\mathbb{C}^n$, and let $f\colon (X, \underline{0}) \to (\mathbb{C}, 0)$ be a holomorphic function. Let $\{S_\alpha\}_{\alpha\in A}$ be an analytic Whitney stratification of X and let $\{T_\gamma\}_{\gamma\in G}$ be an analytic Whitney stratification of $\mathbb{C}$. We say that the stratifications $\{S_\alpha\}_{\alpha\in A}$ and $\{T_\gamma\}_{\gamma\in G}$ give a *Whitney stratification of f* if for every $\alpha \in A$ there exists $\gamma \in G$ such that f induces a submersion from S_α to T_γ.

Consider the triple (y, S_α, S_β), with $y \in S_\alpha \subset \bar{S}_\beta$. Set $f^\alpha_y = f|^{-1}_{S_\alpha}(f(y))$, the fibre of $f|_{S_\alpha}$ which contains y.

We say that a triple (y, S_α, S_β) satisfies the *Thom* (a_f) *condition* if for every sequence $\{x_n\} \subset S_\beta$ converging in $\mathbb{C}^n$ to $y \in S_\alpha$, one has (when the limit exists):

$$\lim_{n\to+\infty} \delta(T_y f^\alpha_y, T_{x_n} f^\beta_{x_n}) = 0.$$

The triple satisfies the *Strict Thom* (w_f) *condition* if there exists a neighbourhood $\mathcal{V}_y$ of $y \in S_\alpha$ in $\mathbb{C}^n$ and a constant $D > 0$ such that for all $y' \in \mathcal{V}_y \cap S_\alpha$ and $x \in \mathcal{V}_y \cap S_\beta$,

$$\delta(T_{y'} f^\alpha_{y'}, T_x f^\beta_x) \le D\|y' - x\| \,.$$

Definition 1.7. We say that the stratification satisfies *Thom's* (a_f) *condition* (respectively (w_f) *condition*) if every triple (y, S_α, S_β) satisfies condition (a_f) (respectively (w_f)).

Definition 1.8. We say that the map f on X has the *Thom property* (respectively the *strict Thom property*) if there is a Whitney stratification of f that satisfies *Thom's* (a_f) *condition* (respectively the (w_f) condition).

Thom property for complex analytic maps was proved by Hironaka in [13, §5 Cor. 1]. The (w_f) property was proved in [24] for the case X smooth (and complex) and in [3, Thm. 4.3.2] for the general complex analytic case. The corresponding statement is false in general for real analytic maps.

Remark 1.9. Let $X \subset U \subset \mathbb{C}^n$ be a complex analytic subset of an open set U of $\mathbb{C}^n$. Let $f\colon (X, \underline{0}) \to (\mathbb{C}, 0)$ be holomorphic. Let $\{S_\alpha\}$ and $\{T_\gamma\}$ be complex analytic Whitney stratifications of X and $\mathbb{C}$ respectively, which define a stratification of f. By [3, Rmk. 4.1.2, Thm. 4.2.1, Thm. 4.3.2] one has that this stratification of f satisfies conditions (a_f) and (w_f).

1.4. Stratified rugose vector fields

Let $X \subset \mathbb{F}^n$ be equipped with a stratification $\{S_\alpha\}$ and $f\colon X \to \mathbb{R}$ an analytic function. The function f is a *rugose function* if for every stratum S_α the restriction $f|_{S_\alpha}$ is of class C^∞ and if for every $y \in S_\alpha$ there exists a neighbourhood V of y and a constant $C \in \mathbb{R}^*_+$, such that, for every $y' \in V \cap S_\alpha$ and every $x \in V \cap X$ we have

$$|f(y') - f(x)| \le C\|y' - x\|.$$

Notice that a rugose function is continuous. A vector valued map is *rugose* if every coordinate function is rugose. A vector bundle F on X is *rugose* if the changes of charts are rugose.

Definition 1.10. A *rugose vector bundle F on X tangent to the stratification* $\{S_\alpha\}$ is a vector bundle on X such that, for every stratum S_α there is an injection $i_\alpha\colon F|_{S_\alpha} \hookrightarrow TS_\alpha$ and if $i\colon X \to \mathbb{F}^n$ is the inclusion, the vector bundle morphism $F \to i^*T\mathbb{F}^n|_X$ induced by i and the i_α is rugose.

A *stratified vector field* v on X is a section of the tangent bundle $T\mathbb{F}^n|_X$, such that at each $x \in X$, the vector $v(x)$ is tangent to the stratum that contains x.

A stratified vector field v is called *rugose* near $y \in S_\alpha$, where S_α is a stratum of X, when there exists a neighbourhood $\mathcal{W}_y$ of y in $\mathbb{F}^n$ and a constant $K > 0$, such that

$$\|v(y') - v(x)\| \leq K\|y' - x\|, \tag{4}$$

for every $y' \in \mathcal{W}_y \cap S_\alpha$ and every $x \in \mathcal{W}_y \cap S_\beta$, with $S_\alpha \subset \bar{S}_\beta$.

Rugose vector fields play a key-role in Verdier's proof of the Thom-Mather isotopy theorems (see [36, §4]).

The following result of Verdier is important for this article. We include a slight modification of the proof given in [36, Prop. (4.6)] which is more suitable for our purposes.

Proposition 1.11 ([36, Prop. (4.6)])**.** *Let X be a real analytic space and A a locally closed subset of X which is union of strata for some Whitney-strong stratification $\{S_\alpha\}$. Let Y be a non-singular real analytic space, $g\colon X \to Y$ a real analytic map whose restriction to each stratum is a submersion, and η a C^∞ vector field on Y. Then there exists a rugose stratified vector field ξ on A that lifts η, i.e., for each $x \in A$ one has $dg(\xi(x)) = \eta(g(x))$.*

Proof. Using a rugose partition of unity the problem is local on A. Let $x \in A$, by [36, Cor. (4.5)] there exists a neighbourhood V of x and a rugose vector bundle F on $V \cap A$ tangent to the stratification $\{S_\alpha \cap V\}$, which induces on the stratum $S_x \cap V$ containing x the tangent bundle TS_x. The map f induces a rugose vector bundle morphism $Tf\colon F \to f^*TY$. Since f is a submersion on each stratum, Tf is surjective in x, and therefore surjective on a neighbourhood of x. Let K be the kernel of F which is the *vertical* subbundle of F. Let H be a rugose *horizontal* subbundle of F, that is, such that $F = K \oplus H$. One way to construct such H is to endow F with a rugose metric and take $H = K^\perp$. Hence Tf induces a rugose isomorphism from H to f^*TY. The inverse image of $f^*\eta$ by this isomorphism is a desired rugose stratified vector field. □

Remark 1.12. Notice that in the proof of Proposition 1.11, different choices of horizontal subbundles H of F give rise to different liftings of η.

2. The uniform conical structure

Let $\langle \cdot\,, \cdot \rangle$ be the Hermitian inner product on $\mathbb{C}^n$. We consider $\mathbb{C}^n$ as a $2n$-dimensional Euclidean real vector space defining the Euclidean inner product as the real part $\Re\langle \cdot\,, \cdot \rangle$. Throughout this article, X is a complex analytic subset of an open set U around the origin $\underline{0}$ of $\mathbb{C}^n$. Let $f\colon (X, \underline{0}) \to (\mathbb{C}, 0)$ be a non-constant holomorphic map and we equip X with a Whitney stratification adapted to $V = f^{-1}(0)$. We let $\mathbb{B}$ be a ball centred at $\underline{0}$ of sufficiently small radius, so that f has no critical points in $\mathbb{B} \setminus V$, each stratum in $X \cap \mathbb{B}$ has $\underline{0}$ in its closure and meets transversally every sphere in $\mathbb{B}$ centred at $\underline{0}$.

Recall that a point $x \in X \cap \mathbb{B}$ is a *critical point of* f, in the stratified sense, if x is a critical point of f restricted to the stratum which contains x (see for instance [10, Part 1 §2.1] for more on this subject). That is, if $\tilde{f}$ is an extension of f to U, then the kernel of $d\tilde{f}(x)$ contains the space tangent to the stratum. One has the analogous definition in the real analytic category (see [10]).

2.1. The canonical pencil of a holomorphic map

Given f, we associate to it a 1-parameter family of real valued functions as follows. For each $\theta \in [0, \pi)$, consider the real line $\mathcal{L}_\theta \subset \mathbb{C}$ passing through the origin with an angle θ with respect to the real axis, measured in the usual way. Let $\mathcal{L}_\theta^\perp$ be the line orthogonal to $\mathcal{L}_\theta$ and let $\pi_\theta \colon \mathbb{C} \to \mathcal{L}_\theta^\perp$ be the orthogonal projection. Set $h_\theta = \pi_\theta \circ f$, so that h_0 and $h_{\frac{\pi}{2}}$ are, respectively, the imaginary and real parts of f. Hence $\{h_\theta\}$ is a 1-parameter family of real analytic functions and if we set $X_\theta = h_\theta^{-1}(0)$, then each X_θ is a real hypersurface.

The first two lemmas below are exercises and we leave the proofs to the reader. They prove part of statement **i)** of Theorem 1.

Lemma 2.1. *The singular points of X_θ are:*

$$\operatorname{Sing} X_\theta = \operatorname{Sing} V \cup (X_\theta \cap \operatorname{Sing} X).$$

Lemma 2.2. *One has $X \cap \mathbb{B} = \cup X_\theta$ and*

$$V = \cap X_\theta = X_{\theta_1} \cap X_{\theta_2},$$

for each pair $\theta_1 \neq \theta_2 \pmod{\pi}$.

Remark 2.3. We notice that each X_θ is naturally the union of three sets: the points $x \in X \cap \mathbb{B}$ such that $f(x) = 0$, *i.e.*, $x \in V$, and the points $x \in X \cap \mathbb{B}$ such that $f(x)$ is in one of the two half lines of $\mathcal{L}_\theta \setminus \{0\}$. Write this as:

$$X_\theta = E_\theta \cup V \cup E_{\theta+\pi}\,.$$

Similarly, if $\mathbb{S} = \partial\mathbb{B}$ one has:

$$(X_\theta \cap \mathbb{S}) = (E_\theta \cap \mathbb{S}) \cup (V \cap \mathbb{S}) \cup (E_{\theta+\pi} \cap \mathbb{S})\,. \tag{5}$$

Remark 2.4. Let $\{S_\alpha\}$ be a Whitney stratification of X adapted to V. Since $f \colon (X \cap \mathbb{B}) \setminus V \to \mathbb{C} \setminus \{0\}$ is a submersion on each stratum, then for each stratum $\{S_\alpha\} \subset X \setminus V$, the intersection $S_\alpha \cap X_\theta$ is a differentiable manifold of real codimension 1 in $\{S_\alpha\}$.

The following lemma proves statement **ii)** of Theorem 1.

Lemma 2.5. *There exists $\varepsilon_0 > 0$ such that given a Whitney stratification $\{S_\alpha\}$ of X adapted to V, the intersection of the strata with each X_θ determines a Whitney strong stratification of X_θ such that for each stratum $S_\alpha \neq \{0\}$ and each X_θ one has that $S_\alpha \cap X_\theta$ meets transversally every sphere in the ball $\mathbb{B}_{\varepsilon_0}$ centred at $\underline{0}$.*

Proof. Since $\bar{S}_\alpha \cap X_\theta = \overline{S_\alpha \cap X_\theta}$, it is clear that the partition of X_θ induced by the intersection with the strata $\{S_\alpha\}$ satisfies the frontier condition and defines a stratification of each X_θ. To see that this stratification of X_θ is Whitney-strong, first note that by [36, Rem. (3.7)], $\{(X_\theta \setminus V) \cap S_\alpha\}$ is a Whitney-strong stratification of $X_\theta \setminus V$, hence we just need to check condition (w) for triples $(y, S_\alpha, (S_\beta \cap X_\theta))$, with $S_\alpha \subset V$ and $(S_\beta \cap X_\theta) \subset X_\theta \setminus V$. By Remark 1.9 f satisfies condition (w_f), so there exists a neighbourhood $\mathcal{V}_y$ of $y \in S_\alpha$ in $\mathbb{C}^n$ and a constant $D > 0$ such that for all $y' \in \mathcal{V}_y \cap S_\alpha$ and $x \in \mathcal{V}_y \cap S_\beta$,

$$\delta(T_{y'} S_\alpha, T_x f_x^\beta) \leq D \|y' - x\|.$$

Since $T_x f_x^\beta \subset T_x(S_\beta \cap X_\theta)$, for every $y' \in \mathcal{V}_y \cap S_\alpha$ and $x \in \mathcal{V}_y \cap (S_\beta \cap X_\theta)$,

$$\delta(T_{y'} S_\alpha, T_x(S_\beta \cap X_\theta)) \leq \delta(T_{y'} S_\alpha, T_x f_x^\beta) \leq D \|y' - x\|.$$

Therefore the triple $(y, S_\alpha, (S_\beta \cap X_\theta))$ satisfies condition (w). We claim that for each θ, each stratum $S_\alpha \cap X_\theta$ meets transversally every sufficiently small sphere around $\underline{0}$. This is in fact an immediate consequence of [2, Lemma 2.4], which implies the existence of a continuous vector field v on $\mathbb{B}_{\varepsilon_0} \setminus V$ which is tangent to each S_α, tangent to each X_θ and transverse to every sufficiently small sphere around $\underline{0}$. $\square$

2.2. The spherefication map

We now introduce an auxiliary function associated to f, the spherefication, which is very helpful for studying Milnor fibrations.

As in Theorem 2, define $\Phi\colon (X \cap \mathbb{B}_\varepsilon) \setminus V \to \mathbb{S}^1$ by $\Phi(x) = \frac{f(x)}{\|f(x)\|}$. Define the real analytic map $\mathfrak{F}\colon (X \cap \mathbb{B}_\varepsilon) \setminus V \to \mathbb{C} \setminus \{0\}$, by

$$\mathfrak{F}(x) = \|x\| \Phi(x) = \|x\| \, \frac{f(x)}{\|f(x)\|} \,. \tag{6}$$

Notice that from the definition we have:

$$\Phi = \frac{\mathfrak{F}(x)}{\|\mathfrak{F}(x)\|} = \frac{f(x)}{\|f(x)\|}. \tag{7}$$

Also notice that given $z \in \mathbb{C} \setminus \{0\}$ with $\theta = \arg z$, the fibre $\mathfrak{F}^{-1}(z)$ is the intersection of X_θ with the sphere $\mathbb{S}_{|z|}$ of radius $|z|$ centred at $\underline{0}$, and $\mathfrak{F}$ carries $\mathbb{S}_{|z|} \setminus V$ into the circle around $0 \in \mathbb{R}^2$ of radius $|z|$. This motivates the following definition:

Definition 2.6. The analytic map $\mathfrak{F}$ is called the *spherefication* of f.

Lemma 2.7. *Let $\varepsilon_0 > 0$ as in Lemma 2.5 and let $\{S_\alpha\}$ be a Whitney stratification adapted to V. Then the spherefication map $\mathfrak{F}$ is a submersion on each stratum.*

Proof. Let $x \in S_\alpha$ and $\theta = \arg f(x)$, then $x \in S_\alpha \cap X_\theta$ and by Lemma 2.5, $S_\alpha \cap X_\theta$ is transverse to the sphere $\mathbb{S}_{\|x\|}$ of radius $\|x\|$. Hence there is a vector $\mu \in T_x(S_\alpha \cap \mathbb{S}_{\|x\|})$ such that $d_x\mathfrak{F}|_{S_\alpha}(\mu)$ is a non-zero vector in $T_{\mathfrak{F}(x)}\mathbb{S}^1_{\|x\|}$. On the other hand, since S_α and $\mathbb{S}_{\|x\|}$ are transverse, there is a vector $\nu \in T_xS_\alpha$ transverse to $\mathbb{S}_{\|x\|}$, and since the fibre of $\mathfrak{F}$ through x is contained in $\mathbb{S}_{\|x\|}$, we have that $d_x\mathfrak{F}|_{S_\alpha}(\nu)$ is a non-zero vector transverse to $\mathbb{S}^1_{\|x\|}$. □

Now we prove statement **iii)** of Theorem 1. For this we use:

Proposition 2.8. *Let $\{S_\alpha\}$ be a Whitney stratification of X adapted to V and for each $\theta \in [0, \pi)$ equip X_θ with the stratification $\{X_\theta \cap S_\alpha\}$ obtained by intersecting X_θ with the strata of $\{S_\alpha\}$. Then for every sufficiently small ball $\mathbb{B}_\varepsilon$ around $\underline{0}$, there exists a stratified, rugose vector field v on $X \cap \mathbb{B}_\varepsilon$ which has the following properties:*

- i) *It is radial, i.e., it is transverse to the intersection of X with all spheres in $\mathbb{B}_\varepsilon$ centred at $\underline{0}$.*
- ii) *It is tangent to the strata of each X_θ.*
- iii) *$v(\underline{0}) = 0$.*

Proof. Let u be the canonical radial vector field on $\mathbb{C}$ given by $u(z) = z$. Using $\mathfrak{F}$, by Proposition 1.11 we can lift u to a stratified rugose vector field $v_{\mathfrak{F}}$ on $(X \cap \mathbb{B}) \setminus V$, which satisfies

$$d\mathfrak{F}_x(v_{\mathfrak{F}}(x)) = u(\mathfrak{F}(x)),$$

for every $x \in (X \cap \mathbb{B}) \setminus V$, where d is the derivative. By the definition of $\mathfrak{F}$, the local flow associated to $v_{\mathfrak{F}}$ is transverse to all spheres in $\mathbb{B}_\varepsilon$ centred at $\underline{0}$, and by construction, its integral paths move along $X_\theta \setminus V$, i.e., it already satisfies conditions i) and ii) and it is rugose.

Recall that on X we have a rugose vector field v_{rad} which is radial at $\underline{0}$, *i.e.*, it is tangent to each stratum and transverse to all spheres around 0, which gives the conical structure of X ([30, Prop. 3.1.7]). The idea to construct the vector field v satisfying the conditions of Proposition 2.8 is to glue the vector field $v_{\mathfrak{F}}$ on $(X \cap \mathbb{B}_\varepsilon) \setminus V$ with the vector field v_{rad} on X in such a way that it keeps satisfying properties i) and ii). The problem is that v_{rad} may not be tangent to the strata of the X_θ, so we must modify it appropriately. We will denote the modified vector field by $\tilde{v}_{rad}$. This will be a rugose vector field defined on $X \cap B_\varepsilon$, which is radial on V, tangent to each stratum $X_\theta \cap S_\alpha$ and such that gluing $\tilde{v}_{rad}$ and $v_{\mathfrak{F}}$ on $X \setminus V$ by a rugose partition of unity, we obtain a vector field v with properties i) to iii). This is constructed as follows.

In [30] the stratified radial vector field v_{rad} is constructed by induction on the dimension of the strata [30, §1.5, Thm. 3.1.5], and it is shown ([30, Prop. 3.1.7]) that it can be assumed to be rugose. We modify v_{rad} to have the desired property at each stage. To start the induction, define $\tilde{v}_{rad}(\underline{0}) = 0$ to get property iii). Now suppose that we have constructed $\tilde{v}_{rad}$ on the strata of dimension less than p and it is rugose, which is always possible by [30, Prop. 3.1.7].

Let S_β be a stratum of dimension p. Extend $\tilde{v}_{rad}$ to a radial stratified rugose vector field v_{rad} on S_β as in [30, Thm. 3.1.5]. If $S_\beta \subset V$, for every $x \in S_\beta$ we define $\tilde{v}_{rad}(x) = v_{rad}(x)$. If $S_\beta \subset X \setminus V$, let $x \in S_\beta$ and denote by f_x^β the fibre of $f|_{S_\beta}$ which contains x, *i.e.*, $f_x^\beta = f|_{S_\beta}^{-1}(f(x))$. Since $f(x)$ is a regular value of $f|_{S_\beta}$, f_x^β is a differentiable submanifold of S_β. Clearly $f_x^\beta \subset X_\theta \cap S_\beta \subset X_\theta \setminus V$ with $\theta = \arg f(x)$. Define the vector $\tilde{v}_{rad}(x)$ by projecting the vector $v_{rad}(x)$ to the tangent space $T_x f_x^\beta \subset T_x(X_\theta \cap S_\beta)$.

We claim that $\tilde{v}_{rad}$ is also rugose. For this, let S_α be a stratum of dimension less than p such that $S_\alpha \subset \bar{S}_\beta$ and let $y \in S_\alpha$. Since the stratification of f satisfies condition (w_f) (see Remark 1.9), there exists a neighbourhood $\mathcal{V}_y$ of y where the following inequality is satisfied.

$$\delta(T_{y'} f_{y'}^\alpha, T_x f_x^\beta) \leq D\|y' - x\|, \tag{8}$$

for all $y' \in \mathcal{V}_y \cap S_\alpha$ and all $x \in \mathcal{V}_y \cap S_\beta$.

On the other hand, since the vector field v_{rad} is rugose, there exists a neighbourhood $\mathcal{W}_y$ of y where the following inequality is satisfied

$$\|v_{rad}(y') - v_{rad}(x)\| \leq K\|y' - x\|, \tag{9}$$

for every $y' \in \mathcal{W}_y \cap S_\alpha$ and every $x \in \mathcal{W}_y \cap S_\beta$.

Let $\mathcal{N}_y$ be an open ball around y such that $\mathcal{N}_y \subset \mathcal{V}_y \cap \mathcal{W}_y$ and set

$$M = \sup_{y' \in \overline{\mathcal{N}_y \cap S_\alpha}} \|v_{rad}(y')\|.$$

Let $y' \in \mathcal{N}_y \cap S_\alpha$ and $x \in \mathcal{N}_y \cap S_\beta$.

Case 1: $S_\beta \subset V$.

Since in V the vector field $\tilde{v}_{rad}$ equals v_{rad} and v_{rad} is rugose, by (9) we have

$$\|\tilde{v}_{rad}(y') - \tilde{v}_{rad}(x)\| = \|v_{rad}(y') - v_{rad}(x)\| \leq K\|y' - x\|.$$

Hence $\tilde{v}_{rad}$ satisfies inequality (4) in $\mathcal{N}_y$.

Case 2: $S_\beta \subset X \setminus V$.

Let π be the orthogonal projection of $\mathbb{C}^n$ onto $T_x f_x^\beta$. By (8) and (3) we have that

$$\left\| \frac{v_{rad}(y')}{\|v_{rad}(y')\|} - \pi\left(\frac{v_{rad}(y')}{\|v_{rad}(y')\|}\right) \right\| \leq \delta(T_{y'} f_{y'}^\alpha, T_x f_x^\beta) \leq D\|y' - x\|.$$

Hence

$$\|v_{rad}(y') - \pi(v_{rad}(y'))\| \leq \|v_{rad}(y')\| D\|y' - x\| \leq MD\|y' - x\|. \tag{10}$$

On the other hand, since π is an orthogonal projection, by (9) one has:

$$\|\pi(v_{rad}(y')) - \pi(v_{rad}(x))\| \leq \|v_{rad}(y') - v_{rad}(x)\| \leq K\|y' - x\|. \tag{11}$$

Therefore using (10) and (11) we get:

$$\begin{aligned}\|\tilde{v}_{rad}(y') - \tilde{v}_{rad}(x)\| &= \|v_{rad}(y') - \pi(v_{rad}(x))\| \\ &\leq \|v_{rad}(y') - \pi(v_{rad}(y'))\| + \|\pi(v_{rad}(y')) - \pi(v_{rad}(x))\| \\ &\leq (MD + K)\|y' - x\|.\end{aligned}$$

Hence $\tilde{v}_{rad}$ satisfies inequality (4) in $\mathcal{N}_y$, proving that $\tilde{v}_{rad}$ is rugose and therefore integrable [36, Prop. (4.8)].

Notice that when we modify the radial vector field v_{rad} to obtain $\tilde{v}_{rad}$, it may happen that $\tilde{v}_{rad}(x)$ is no longer transverse to the sphere, this is the case if the fibre through x is tangent to the sphere; it may even happen that $\tilde{v}_{rad}(x)$ vanishes.

Gluing $\tilde{v}_{rad}$ and $v_{\mathfrak{F}}$ on $X \setminus V$ by a rugose partition of unity we obtain a vector field v defined on all of $\mathbb{B}_\varepsilon$ with the desired properties. □

Proof of **iii)** *in Theorem 1.* Recall that $\langle \cdot, \cdot \rangle$ is the Hermitian inner product on $\mathbb{C}^n$ and that we consider $\mathbb{C}^n$ as a $2n$-dimensional Euclidean real vector space defining the Euclidean inner product as the real part $\Re\langle \cdot, \cdot \rangle$. Let $r\colon \mathbb{C}^n \to \mathbb{R}$ be the real analytic function defined by $r(x) = \|x\|^2$. The *real* gradient of r is given by $\operatorname{grad}_{\mathbb{R}} r(x) = 2x$, so the chain rule for the derivative of r along a path $x = p(t)$ takes the form

$$\frac{dr(p(t))}{dt} = \Re\Big\langle \frac{dp}{dt}, 2x \Big\rangle. \tag{12}$$

We proceed as in the proof of [22, Thm. 2.10]: let v be the vector field constructed in Proposition 2.8. By property i) v is a radial vector field pointing away from $\underline{0}$, thus we have that

$$\Re\langle v(x), x \rangle > 0.$$

Normalise v by setting

$$\hat{v}(x) = v(x)/\Re\langle v(x), 2x \rangle.$$

Let $x = p(t)$ be a solution of the differential equation

$$\frac{dp(t)}{dt} = \hat{v}(p(t)). \tag{13}$$

Then by (12) we have that

$$\frac{dr}{dt}(x) = \Re\langle \hat{v}(x), 2x \rangle = 1.$$

Therefore $r(p(t)) = t + \text{constant}$. Substracting a constant from the parameter t if necessary, we may suppose that

$$r(p(t)) = \|p(t)\|^2 = t.$$

By [22, p. 20], the solution $p(t)$ can be extended through the interval $(0, \varepsilon^2]$.

For every $a \in X \cap \mathbb{S}_\varepsilon$ let $p(t) = P(a,t)$ be the unique solution of (13) which satisfies the initial condition $p(\varepsilon^2) = P(a, \varepsilon^2) = a$. Clearly this function P gives a *rugose homeomorphism* from the product $L_X \times (0, \varepsilon^2]$ onto $(X \setminus \{\underline{0}\}) \cap \mathbb{B}_\varepsilon$.

Since $\hat{v}$ is a stratified vector field and V is a finite union of strata, any solution curve which touches a stratum $S_\alpha \subset V$ must be contained in $S_\alpha \subset V$. Therefore, P is in fact a rugose homeomorphism of pairs from $(L_X \times (0, \varepsilon^2], L_f \times (0, \varepsilon^2])$ onto $((X \setminus \{\underline{0}\}) \cap \mathbb{B}_\varepsilon, (V \setminus \{\underline{0}\}) \cap \mathbb{B}_\varepsilon)$.

Furthermore, since the vector field $v(x)$ is tangent to the stratum $X_\theta \cap S_\alpha$ of $X_\theta \setminus V$ for all $x \in X \cap S_\alpha$, every solution curve which touches $X_\theta \cap S_\alpha$ must be contained in $X_\theta \cap S_\alpha \subset X_\theta \setminus V$. Hence P restricts to a rugose homeomorphism $(X_\theta \cap \mathbb{S}_\varepsilon) \times (0, \varepsilon^2] \cong ((X_\theta \setminus \{\underline{0}\}) \cap \mathbb{B}_\varepsilon)$ for every $\theta \in [0, \pi)$.

Finally, note that $P(a,t)$ tends uniformly to $\underline{0}$ as $t \to 0$. Therefore the correspondence

$$ta \to P(a,t), \quad 0 < t \leq 1, \quad a \in X \cap \mathbb{S}_\varepsilon,$$

extends uniquely to a homeomorphism from $\mathrm{Cone}(L_X)$ to $(X \cap \mathbb{B}_\varepsilon)$ and we arrive to statement **iii)** in Theorem 1. □

Remark 2.9. Notice that if $X \setminus \underline{0}$ is non-singular, the proof above leads to a smooth vector field having properties i)-iii) in Proposition 2.8. This yields to a *diffeomorphism* between $X \setminus \underline{0}$ and the cylinder $L_X \times (-\infty, 0]$, where L_X is the link of X, inducing for each θ a diffeomorphism $X_\theta \setminus \underline{0} \cong L_{X_\theta} \times (-\infty, 0]$.

Let us denote by $\mathcal{X}_{(X \setminus V) \cap \mathbb{B}_\varepsilon}$ the decomposition of $(X \setminus V) \cap \mathbb{B}_\varepsilon$ as the union of all $(X_\theta \setminus V) \cap \mathbb{B}_\varepsilon$; similarly, denote by $\mathcal{X}_{L_X \setminus L_f}$ the decomposition of $L_X \setminus L_f$ as the union of all $(X_\theta \cap \partial\mathbb{B}_\varepsilon) \setminus L_f$.

The following is an immediate consequence of the proof of Proposition 2.8.

Corollary 2.10. *The decomposition $\mathcal{X}_{(X \setminus V) \cap \mathbb{B}_\varepsilon}$ is homeomorphic to the cylinder $(\mathcal{X}_{L_X \setminus L_f}) \times (-\infty, 0]$. Furthermore, if X is non-singular away from V, then the above homeomorphism can be taken to be a diffeomorphism.*

3. The Fibration theorems

In this section we prove Theorems 2 and 3. The proof of Theorem 3 is based on the results in Section 2 together with Lemmas 3.1 and 3.2 below, which are extensions of Milnor's Lemmas 4.3 and 4.4 in [21] to the case of maps defined on singular varieties.

Throughout this section let $\tilde{f} \colon U \to \mathbb{C}$ be an analytic extension of f to an open neighbourhood U of $\underline{0}$ in $\mathbb{C}^n$.

Lemma 3.1. *Let S_α be a stratum not contained in V, $x \in S_\alpha$ and let π_{α_x} be the orthogonal projection of $\mathbb{C}^n$ onto $T_x S_\alpha$. Let $p\colon [0,\varepsilon) \to S_\alpha$ be a real analytic path with $p(0) = \underline{0}$ such that, for each $t > 0$, the number $f\big(p(t)\big)$ is non-zero and the vector $\pi_{\alpha_{p(t)}}\big(\operatorname{grad} \log \tilde{f}\big(p(t)\big)\big)$ is a complex multiple $\lambda(t)\pi_{\alpha_{p(t)}}(p(t))$. Then the argument of the complex number $\lambda(t)$ tends to zero as $t \to 0$. In other words $\lambda(t)$ is non-zero for small positive values of t and $\lim_{t\to 0} \frac{\lambda(t)}{|\lambda(t)|} = 1$.*

Proof. We follow the proof of [21, Lemma 4.4]. Consider the Taylor expansions

$$\begin{aligned} p(t) &= \mathbf{a}t^k + \mathbf{a}_1 t^{k+1} + \mathbf{a}_2 t^{k+2} + \dots, \\ f\big(p(t)\big) &= bt^l + b_1 t^{l+1} + b_2 t^{l+2} + \dots, \\ \operatorname{grad} \tilde{f}\big(p(t)\big) &= \mathbf{c}t^m + \mathbf{c}_1 t^{m+1} + \mathbf{c}_2 t^{m+2} + \cdots, \end{aligned}$$

where the leading coefficients $\mathbf{a}$, b and $\mathbf{c}$ are non-zero. (The identity $\frac{df}{dt} = \langle \frac{dp}{dt}, \operatorname{grad} \tilde{f} \rangle$ shows that $\operatorname{grad} \tilde{f}(p(t))$ cannot be identically zero.) The leading exponents k, l and m are integers with $k \geq 1$, $l \geq 1$ and $m \geq 0$. The series are all convergent say for $|t| < \varepsilon'$. To simplify notation, set $\pi_{\alpha_t} = \pi_{\alpha_{p(t)}}$. For each $t > 0$ we have

$$\begin{aligned} \pi_{\alpha_t}\Big(\operatorname{grad} \log \tilde{f}\big(p(t)\big)\Big) &= \lambda(t)\pi_{\alpha_t}\big(p(t)\big), \\ \pi_{\alpha_t}\Big(\operatorname{grad} \tilde{f}\big(p(t)\big)\Big) &= \lambda(t)\pi_{\alpha_t}\big(p(t)\big)\bar{f}\big(p(t)\big), \end{aligned}$$

The vector $\pi_{\alpha_t}\Big(\operatorname{grad} \tilde{f}\big(p(t)\big)\Big)$ is non-zero since f is a submersion on S_α. Hence there exists a smallest s such that $\pi_{\alpha_t}(\mathbf{c}_s) \neq 0$. On the other hand, the vector $\pi_{\alpha_t}\big(p(t)\big)$ is non-zero since S_α is transverse to all the spheres centred at $\underline{0}$ of radius less than ε. Hence, there exists a smallest r such that $\pi_{\alpha_t}(\mathbf{a}_r) \neq 0$. Therefore

$$\begin{aligned} \pi_{\alpha_t}(\mathbf{c}t^m + \mathbf{c}_1 t^{m+1} + \cdots) &= \lambda(t)\pi_{\alpha_t}(\mathbf{a}t^k + \mathbf{a}_1 t^{k+1} + \dots)(\bar{b}t^l + \bar{b}_1 t^{l+1} + \dots) \\ \pi_{\alpha_t}(\mathbf{c}_s)t^{m+s} + \dots &= \lambda(t)\big(\pi_{\alpha_t}(\mathbf{a}_r)t^{k+r} + \dots\big)\big(\bar{b}t^l + \dots\big) \\ \pi_{\alpha_t}(\mathbf{c}_s)t^{m+s} + \dots &= \lambda(t)\big(\pi_{\alpha_t}(\mathbf{a}_r)\bar{b}t^{k+r+l} + \dots\big). \end{aligned}$$

Comparing corresponding components of these two vector valued functions, we see that $\lambda(t)$ is a quotient of real analytic functions, and therefore it has a Laurent expansion of the form

$$\lambda(t) = \lambda_0 t^{m+s-k-r-l}(1 + d_1 t + d_2 t^2 + \dots).$$

Furthermore the leading coefficients must satisfy the equation

$$\pi_{\alpha_t}(\mathbf{c}_s) = \lambda_0 \pi_{\alpha_t}(\mathbf{a}_r)\bar{b}.$$

Substituting this equation in the power series expansion of the identity

$$\frac{df(p(t))}{dt} = \Big\langle \frac{dp}{dt}, \operatorname{grad} \tilde{f}\big(p(t)\big) \Big\rangle,$$

and noting that since $p(t) \in S_\alpha$, $t \in [0, \varepsilon)$, we have that $\frac{dp}{dt} \in T_{p(t)}S_\alpha$ and therefore $\pi_{\alpha_t}(\frac{dp}{dt}) = \frac{dp}{dt}$. Thus we obtain

$$\frac{df\big(p(t)\big)}{dt} = \Big\langle \pi_{\alpha_t}\Big(\frac{dp}{dt}\Big), \pi_{\alpha_t}\Big(\operatorname{grad} \tilde{f}\big(p(t)\big)\Big)\Big\rangle.$$

Therefore

$$\begin{aligned}
(lbt^{l-1} + \dots) &= \Big\langle \pi_{\alpha_t}(k\mathbf{a}t^{k-1} + \dots), \pi_{\alpha_t}(\mathbf{c}t^m + \dots)\Big\rangle, \\
&= \Big\langle k\pi_{\alpha_t}(\mathbf{a}_r)t^{k+r-1} + \dots, \pi_{\alpha_t}(\mathbf{c}_s)t^{m+s} + \dots \Big\rangle \\
&= \Big\langle k\pi_{\alpha_t}(\mathbf{a}_r)t^{k+r-1} + \dots, \lambda_0\pi_{\alpha_t}(\mathbf{a}_r)\bar{b}t^{m+s} + \dots \Big\rangle \\
&= k\|\pi_{\alpha_t}(\mathbf{a}_r)\|^2\bar{\lambda_0}bt^{k+r+m+s-1} + \dots .
\end{aligned}$$

Comparing the leading coefficients it follows that

$$l = k\|\pi_{\alpha_t}(\mathbf{a}_r)\|^2\bar{\lambda_0}$$

which proves that λ_0 is a positive real number. Therefore

$$\lim_{t\to 0} \arg \lambda(t) = 0,$$

which completes the proof of Lemma 3.1. □

Lemma 3.2. *Let X be an analytic subset of an open neighbourhood U of the origin $\underline{0}$ in $\mathbb{C}^n$. Let $f\colon (X, \underline{0}) \to (\mathbb{C}, 0)$ be holomorphic and set $V = f^{-1}(0)$. Let $\tilde{f}\colon U \to \mathbb{C}$ be an analytic extension of f to U. Let S_α be a stratum not contained in V, $x \in S_\alpha$ and let π_{α_x} be the orthogonal projection of $\mathbb{C}^n$ onto T_xS_α. There exists a number $\varepsilon_0 > 0$ so that, for every $x \in S_\alpha$ with $\|x\| \le \varepsilon_0$, the two vectors $\pi_{\alpha_x}(x)$ and $\pi_{\alpha_x}(\operatorname{grad}\log\tilde{f}(x))$ are either linearly independent over the complex numbers or else*

$$\pi_{\alpha_x}\big(\operatorname{grad}\log\tilde{f}(x)\big) = \lambda\pi_{\alpha_x}(x)$$

where λ is a non-zero complex number whose argument has absolute value less than $\pi/4$.

Proof. The proof is the same as in [21, Lemma 4.3] defining

$$W = \{\, z \in S_\alpha \mid \pi_{\alpha_z}\big(\operatorname{grad}\log\tilde{f}(z)\big) = \mu\pi_{\alpha_z}(z),\, \mu \in \mathbb{C}\,\},$$

and using the *analytic* curve selection lemma [4, Prop. 2.2]. □

As in Milnor's case, we have the following corollary as an immediate consequence.

Corollary 3.3. *Let S_α be a stratum not contained in V. For every $x \in S_\alpha$ which is sufficiently close to the origin, the two vectors $\pi_{\alpha_x}(x)$ and $\pi_{\alpha_x}\big(i\operatorname{grad}\log\tilde{f}(x)\big)$ are linearly independent over $\mathbb{R}$.*

The vector $\pi_{\alpha_x}(x)$ is the normal vector in S_α of the codimension 1 submanifold $\mathbb{S}_{\|x\|} \cap S_\alpha$ and the vector $\pi_{\alpha_x}\big(i \operatorname{grad} \log \tilde{f}(x)\big)$ is the normal vector in S_α of the codimension 1 submanifold $S_\alpha \cap X_\theta$. Therefore Corollary 3.3 gives another proof of the second statement of Theorem 1-**ii)**, that the intersection $S_\alpha \cap X_\theta$ meets transversally every sphere in $\mathbb{B}_\varepsilon$ centred at $\underline{0}$.

Proposition 3.4. *There exists a complete, stratified, rugose, vector field w on $(X \cap \mathbb{B}_\varepsilon) \setminus V$, tangent to all the spheres in $\mathbb{B}_\varepsilon$ centred at $\underline{0}$, and whose orbits are transverse to the $X_\theta \setminus V$ and permute them: for each fixed time t, the flow carries each $X_\theta \setminus V$ into $X_{\theta+t} \setminus V$, where the angle $\theta + t$ must be taken modulo π. In particular, for $t = \pi$ the flow interchanges the two halves of $X_\theta \setminus V$.*

Proof. Let $\varepsilon_0 > 0$ as in Lemma 2.5. By Lemma 2.7, the map

$$\mathfrak{F} \colon (X \cap \mathbb{B}_\varepsilon) \setminus V \to \mathbb{C} \setminus \{0\}$$

is a submersion on each stratum $S_\alpha \subset \mathbb{B}_{\varepsilon_0}$.

Consider the vector field $\bar{w}$ on $\mathbb{C} \setminus \{0\}$ given by $\bar{w}(z) = iz$, which is tangent to all the circles $\mathbb{S}^1_\eta \subset \mathbb{C}$. By Proposition 1.11 we can lift $\bar{w}$ using $\mathfrak{F}$, to a stratified rugose vector field w on $(X \cap \mathbb{B}_{\varepsilon_0}) \setminus V$. By [36, Prop. (4.8)] this vector field is integrable and since $d\mathfrak{F}_x(w(x)) = \bar{w}(\mathfrak{F}(x))$, the integral curve $p(t)$ of w is sent by $\mathfrak{F}$ to the circle $\mathbb{S}^1_{\|x\|} \subset \mathbb{C}$ of radius $\|x\|$. Thus, $p(t)$ is transverse to X_θ with $\theta = \arg \mathfrak{F}(x)$. On the other hand, by the definition of $\mathfrak{F}$, we have that $p(t)$ lies in the sphere $\mathbb{S}_{\|x\|} \subset \mathbb{C}^n$ and therefore w is tangent to all the spheres in $\mathbb{B}_{\varepsilon_0}$. The solution $p(t)$ certainly exists locally and can be extended over some maximal open interval of $\mathbb{R}$.

Since $(X \cap \mathbb{B}_\varepsilon) \setminus V$ is not compact, we have to guarantee that $p(t)$ cannot tend to V as t tends to some finite limit t_0, that is, that w is a complete vector field. The lifting w of the vector field $\bar{w}$ is not unique and it depends on the choice of a horizontal subbundle H of the rugose vector field F tangent to the stratification $\{S_\alpha\}$ which induces the tangent bundle TS_α on a stratum $S_\alpha \cap V$(see Remark 1.12). Hence we have to choose an appropriate horizontal subbundle H to ensure that the lifting w is complete. We do this as follows.

Let $\tilde{f} \colon U \to \mathbb{C}$ be an analytic extension of f to an open neighbourhood U of $\underline{0}$ in $\mathbb{C}^n$. Let $\tilde{V} = \tilde{f}^{-1}(0)$, $\tilde{h}_\theta = \pi_\theta \circ \tilde{f}$ and define $\tilde{X}_\theta = \tilde{h}_\theta^{-1}(0)$. Following [22, §4], since $\tilde{f} = |\tilde{f}(x)| e^{i\theta(x)}$, we have that $\log \tilde{f}(x) = \log|\tilde{f}(x)| + i\theta(x)$, thus $\theta(x) = \Re(-i \log \tilde{f}(x))$. Differentiating along a curve $x = p(t)$ we have that

$$\frac{d\theta(p(t))}{dt} = \Re\Big\langle \frac{dp}{dt}, i \operatorname{grad} \log \tilde{f}(x) \Big\rangle. \tag{14}$$

Therefore, the vector $i \operatorname{grad} \log \tilde{f}(x)$ is the *real* gradient of the real analytic function θ. Since the $\tilde{X}_\theta$'s are the fibres of θ the vector $i \operatorname{grad} \log \tilde{f}(x)$ is *normal* to $\tilde{X}_{\theta(x)}$.

On the other hand, $\log|\tilde{f}(x)| = \Re(\log \tilde{f}(x))$ and differentiating along a curve $x = p(t)$ we have that

$$\frac{d \log|\tilde{f}(x)|}{dt} = \Re\Big\langle \frac{dp}{dt}, \operatorname{grad} \log \tilde{f}(x) \Big\rangle. \tag{15}$$

Hence the vector $\operatorname{grad}\log\tilde{f}(x)$ is the *real* gradient of the real analytic function $\log|\tilde{f}(x)|$. Since the Milnor tubes are the fibres of $\log|\tilde{f}(x)|$, the vector $\operatorname{grad}\log\tilde{f}(x)$ is *normal* to the Milnor tube $N(\varepsilon,|\tilde{f}(x)|)$.

We have that $X_\theta = X\cap\tilde{X}_\theta$. Let S_α be a stratum of X not contained in V; notice that $S_\alpha\cap X_\theta = S_\alpha\cap\tilde{X}_\theta$. Consider $x\in S_\alpha\cap X_\theta$ and let $\mathbb{S}_{\|x\|}$ be the sphere of radius $\|x\|$ centred at $\underline{0}$. By the definition of $\mathfrak{F}$ we have that the fibre of $\mathfrak{F}|_{S_\alpha}$ which contains x is given by

$$\mathfrak{F}|_{S_\alpha}^{-1}(\mathfrak{F}(x)) = S_\alpha\cap\mathbb{S}_{\|x\|}\cap X_\theta. \tag{16}$$

We denote this fibre by $\mathfrak{F}_x^\alpha$. To simplify notation set $\mathbb{S}_{\|x\|}^\alpha = S_\alpha\cap\mathbb{S}_{\|x\|}$, $X_\theta^\alpha = S_\alpha\cap X_\theta$ and $N_x^\alpha = S_\alpha\cap N(\varepsilon,|\tilde{f}(x)|)$. Let π_{α_x} be the orthogonal projection of $\mathbb{C}^n$ onto T_xS_α, then the vector $\pi_{\alpha_x}(i\operatorname{grad}\log\tilde{f}(x))$ is *normal* to X_θ^α, the vector $\pi_{\alpha_x}(\operatorname{grad}\log\tilde{f}(x))$ is *normal* to the Milnor tube N_x^α, and the vector $\pi_{\alpha_x}(x)$ is *normal* to $\mathbb{S}_{\|x\|}^\alpha$. Denote by $\operatorname{span}_{\mathbb{R}}\big(\pi_{\alpha_x}(x),\pi_{\alpha_x}(i\operatorname{grad}\log\tilde{f}(x))\big)$ the *real* plane spanned by the vectors $\pi_{\alpha_x}(x)$ and $\pi_{\alpha_x}(i\operatorname{grad}\log\tilde{f}(x))$. By (16) we have that

$$(T_x\mathfrak{F}_x^\alpha)^\perp = \operatorname{span}_{\mathbb{R}}\big(\pi_{\alpha_x}(x),\pi_{\alpha_x}(i\operatorname{grad}\log\tilde{f}(x))\big),$$

the orthogonal space is taken in the tangent space to the stratum.

As in [22, Lem. 4.6] we have to consider two cases:

Case 1: The vectors $\pi_{\alpha_x}(x)$ and $\pi_{\alpha_x}(\operatorname{grad}\log\tilde{f}(x))$ are linearly independent over $\mathbb{C}$.

We have that $\mathbb{S}_{\|x\|}^\alpha$ and N_x^α are codimension 1 submanifolds of S_α and the vectors $\pi_{\alpha_x}(x)$ and $\pi_{\alpha_x}(\operatorname{grad}\log\tilde{f}(x))$ are respectively normal to them. Since these vectors are linearly independent over $\mathbb{R}$, we have that $\mathbb{S}_{\|x\|}^\alpha$ and N_x^α are transverse. Let M_x^α be their intersection, which is a submanifold of $\mathbb{S}_{\|x\|}^\alpha$ of codimension 1. The fibre $\mathfrak{F}_x^\alpha$ is also a codimension 1 submanifold of $\mathbb{S}_{\|x\|}^\alpha$.

Claim: The manifolds M_x^α and $\mathfrak{F}_x^\alpha$ are transverse in $\mathbb{S}_{\|x\|}^\alpha$.

Proof of Claim. Suppose they are not transverse, then $T_xM_x^\alpha = T_x\mathfrak{F}_x^\alpha$, hence $(T_xM_x^\alpha)^\perp = (T_x\mathfrak{F}_x^\alpha)^\perp$. But

$$(T_xM_x^\alpha)^\perp = \operatorname{span}_{\mathbb{R}}\big(\pi_{\alpha_x}(x),\pi_{\alpha_x}(\operatorname{grad}\log\tilde{f}(x))\big),$$
$$(T_x\mathfrak{F}_x^\alpha)^\perp = \operatorname{span}_{\mathbb{R}}\big(\pi_{\alpha_x}(x),\pi_{\alpha_x}(i\operatorname{grad}\log\tilde{f}(x))\big),$$

here all the orthogonal spaces are considered in the tangent space to the stratum T_xS_α. Therefore the vector $\pi_{\alpha_x}(x)$ is in the *real* plane generated by $\pi_{\alpha_x}(\operatorname{grad}\log\tilde{f}(x))$ and $\pi_{\alpha_x}(i\operatorname{grad}\log\tilde{f}(x))$, this implies that $\pi_{\alpha_x}(x)$ is in the *complex* line generated by $\pi_{\alpha_x}(\operatorname{grad}\log\tilde{f}(x))$, which contradicts our original assumption. ∎

Since M_x^α and $\mathfrak{F}_x^\alpha$ are transverse in $\mathbb{S}_{\|x\|}^\alpha$ there is a *unique* direction in $T_xM_x^\alpha$ which is not in $T_x\mathfrak{F}_x^\alpha$ and is orthogonal to $T_x(M_x^\alpha\cap\mathfrak{F}_x^\alpha)$. Let $\nu(x)$ be a unit

vector in this direction, then the *real* plane $H_x = \mathrm{span}_{\mathbb{R}}\big(\nu(x), \pi_{\alpha_x}(x)\big)$ is complementary to $T_x\mathfrak{F}^\alpha_x$ in T_xS_α, that is

$$T_xS_\alpha = T_x\mathfrak{F}^\alpha_x \oplus H_x.$$

Let W be a neighbourhood of x where there is a rugose vector bundle F on $W \cap X$ tangent to the stratification $\{S_\alpha \cap W\}$ which induces the tangent bundle TS_α on the stratum $S_\alpha \cap W$ [36, Cor. (4.5)]. Consider the rugose vector bundle morphism $T\mathfrak{F}\colon F \to \mathfrak{F}^*T\mathbb{C}$ induced by $\mathfrak{F}$ and let K be its kernel. For a point $x \in S_\alpha$ we have that $F_x = T_xS_\alpha$ and the fibre $K_x = T_x\mathfrak{F}^\alpha_x$.

Let $x' \in W$ and let $\pi_{F_{x'}}$ be the orthogonal projection of $\mathbb{C}^n$ onto the fibre $F_{x'}$ of F at x'. Since the vectors $\pi_{\alpha_x}(x)$ and $\pi_{\alpha_x}(\mathrm{grad}\log\tilde{f}(x))$ are linearly independent over $\mathbb{C}$, the vectors x' and $\mathrm{grad}\log\tilde{f}(x')$ are also linearly independent over $\mathbb{C}$ for every x' in a neighbourhood of x which we can assume is also W. Then, by the previous arguments, there exists a unit vector $\tilde{\nu}(x')$ tangent to the intersection $M_{x'}$ of the sphere $\mathbb{S}_{\|x'\|}$ and the Milnor tube $N(\varepsilon, |\tilde{f}(x')|)$ and orthogonal to $M_{x'} \cap \mathfrak{F}_{x'}$ where $\mathfrak{F}_{x'}$ is the fibre of $\mathfrak{F}$ which contains x'. Define $H_{x'} = \pi_{F_{x'}}\Big(\mathrm{span}_{\mathbb{R}}\big(\tilde{\nu}(x'), x'\big)\Big)$. Notice that for $x \in S_\alpha$ we have that $\nu(x) = \pi_{\alpha_x}(\tilde{\nu}(x))$ and $\pi_{F_x} = \pi_{\alpha_x}$, so both definitions of H_x coincide. Since H_x is complementary to K_x in F_x, $H_{x'}$ is also complementary to $K_{x'}$ in $F_{x'}$ for every x' in a neighbourhood of x which we can also assume is W. Hence H is a horizontal subbundle of F and it gives a rugose stratified lifting w of $\bar{w}$ in W.

Let $p(t)$ be an integral curve of the vector field w. Since $d\mathfrak{F}_x(w(x)) = \bar{w}(\mathfrak{F}(x))$, we have that

$$\theta(p(t)) = t + \text{constant},$$

and by (14),

$$\begin{aligned} 1 = \frac{d\theta(p(t))}{dt} &= \Re\langle w(p(t)), i\,\mathrm{grad}\log\tilde{f}(p(t))\rangle \\ &= \Re\langle w(p(t)), \pi_{F_{p(t)}}(i\,\mathrm{grad}\log\tilde{f}(p(t)))\rangle. \end{aligned}$$

In particular we have that

$$\Re\langle w(x), i\,\mathrm{grad}\log\tilde{f}(x)\rangle = \Re\langle w(x), \pi_{F_x}(i\,\mathrm{grad}\log\tilde{f}(x))\rangle = 1. \tag{17}$$

Notice that by definition, $w(x) \in T_x\mathbb{S}^\alpha_{\|x\|} \cap H_x$ and therefore it is a *real* multiple of the vector $\nu(x)$. Since $\nu(x)$ is tangent to the Milnor tube M^α_x and $\pi_{\alpha_x}(\mathrm{grad}\log\tilde{f}(x))$ is normal to M^α_x, we have that

$$\Re\langle w(x), \mathrm{grad}\log\tilde{f}(x)\rangle = \Re\langle w(x), \pi_{F_x}(\mathrm{grad}\log\tilde{f}(x))\rangle = 0. \tag{18}$$

Combining (17) and (18) we have that

$$\langle w(x), i\,\mathrm{grad}\log\tilde{f}(x)\rangle = 1,$$

which implies that

$$|\arg\langle w(x), i\,\mathrm{grad}\log\tilde{f}(x)\rangle| < \frac{\pi}{4}. \tag{19}$$

This condition certainly holds throughout a neighbourhood of x.

Case 2: The vector $\pi_{\alpha_x}(\operatorname{grad}\log\tilde{f}(x))$ is equal to a multiple $\lambda\pi_{\alpha_x}(x)$.

In this case, the *complex* line generated by $\pi_{\alpha_x}(\operatorname{grad}\log\tilde{f}(x))$ is the *real* plane $H_x = \operatorname{span}_{\mathbb{R}}\big(\pi_{\alpha_x}(x), \pi_{\alpha_x}(i\operatorname{grad}\log\tilde{f}(x))\big) = \operatorname{span}_{\mathbb{R}}\big(\pi_{\alpha_x}(ix), \pi_{\alpha_x}(x)\big)$. Since H_x is the orthogonal space in T_xS_α to $T\mathfrak{F}^\alpha_x$, H_x is complementary to $T\mathfrak{F}^\alpha_x$ in T_xS_α.

As before, let W be a neighbourhood of x where there is a rugose vector bundle F on $W \cap X$ tangent to the stratification $\{S_\alpha \cap W\}$ which induces the tangent bundle TS_α on the stratum $S_\alpha \cap W$. Consider the rugose vector bundle morphism $T\mathfrak{F}\colon F \to \mathfrak{F}^*T\mathbb{C}$ induced by $\mathfrak{F}$ and let K be its kernel. For a point $x \in S_\alpha$ we have that $F_x = T_xS_\alpha$ and the fibre $K_x = T_x\mathfrak{F}^\alpha_x$.

Let $x' \in W$ and let $\pi_{F_{x'}}$ be the orthogonal projection of $\mathbb{C}^n$ onto the fibre $F_{x'}$ of F at x'. Define $H_{x'} = \pi_{F_{x'}}\big(\operatorname{span}_{\mathbb{R}}(ix', x')\big)$. For $x \in S_\alpha$ we have that $\pi_{F_x} = \pi_{\alpha_x}$, so both definitions of H_x coincide. Since H_x is complementary to K_x in F_x, $H_{x'}$ is also complementary to $K_{x'}$ in $F_{x'}$ for every x' in a neighbourhood of x which we can also assume is W. Hence H is a horizontal subbundle of F and it gives a rugose stratified lifting w of $\bar{w}$ in W.

Notice that by definition when $x \in S_\alpha$, $w(x) \in T_x\mathbb{S}^\alpha_{\|x\|} \cap H_x$, so there exists $k \in \mathbb{R}$ such that $w(x) = k\pi_\alpha(ix)$. Therefore we have:

$$\begin{aligned}
\langle w(x), i\operatorname{grad}\log\tilde{f}(x)\rangle &= \langle w(x), \pi_{\alpha_x}(i\operatorname{grad}\log\tilde{f}(x))\rangle\\
&= k\langle \pi_{\alpha_x}(ix), \pi_{\alpha_x}(i\operatorname{grad}\log\tilde{f}(x))\rangle\\
&= k\langle \pi_{\alpha_x}(ix), \lambda\pi_{\alpha_x}(ix)\rangle\\
&= k\bar{\lambda}\|\pi_{\alpha_x}(x)\|^2.
\end{aligned}$$

Since $d\mathfrak{F}_x(w(x)) = \bar{w}(\mathfrak{F}(x))$, we have that

$$\Re\langle w(x), i\operatorname{grad}\log\tilde{f}(x)\rangle = \Re\langle w(x), \pi_{\alpha_x}(i\operatorname{grad}\log\tilde{f}(x))\rangle = 1, \tag{20}$$

and by Lemma 3.2 and since k is real

$$|\arg\langle w(x), i\operatorname{grad}\log\tilde{f}(x)\rangle| < \frac{\pi}{4}.$$

Again, this condition holds throughout a neighbourhood of x and using a rugose partition of unity we obtain a global vector field w which lifts $\bar{w}$ and satisfies the following properties:

$$\begin{aligned}
\Re\langle w(x), i\operatorname{grad}\log\tilde{f}(x)\rangle &= 1,\\
|\arg\langle w(x), i\operatorname{grad}\log\tilde{f}(x)\rangle| &< \frac{\pi}{4}.
\end{aligned}$$

Therefore

$$|\Re\langle w(x), \operatorname{grad}\log\tilde{f}(x)\rangle| = |\Im\langle w(x), i\operatorname{grad}\log\tilde{f}(x)\rangle| < 1\,.$$

To guarantee that $p(t)$ cannot tend to V as t tends to some finite limit t_0 is equivalent to insure that $f(p(t))$ cannot tend to zero, or that $\log|f(p(t))|$ cannot tend to $-\infty$, as $t \to t_0$. But we have that

$$\left|\frac{d\log|f(p(t))|}{dt}\right| = \left|\frac{d\Re\log f(p(t))}{dt}\right| = |\Re\langle w(p(t)), \operatorname{grad}\log\tilde{f}(p(t))\rangle| < 1\,.$$

Hence $\log|f(p(t))| < t + \text{constant}$ and therefore $|f(p(t))|$ is bounded away from zero as t tends to any finite limit.

Taking $\varepsilon_0 > \varepsilon > 0$ we have that w satisfies the lemma on $(X \cap \mathbb{B}_\varepsilon) \setminus V$. $\square$

We notice that Proposition 3.4 essentially proves that the map

$$\Psi : (X \cap \mathbb{B}_\varepsilon) \setminus V \to \mathbb{RP}^1$$

in Theorem 2, is the projection map of a fibre bundle (see Section 3.1 below).

Proof of Theorem 3. That the restriction of $\mathfrak{F}$ to each sphere around $\underline{0}$ of radius $\varepsilon' \leq \varepsilon$ is a fibre bundle over the corresponding circle of radius ε', is an immediate consequence of Proposition 3.4. The composition of this restriction with the radial projection of $\mathbb{S}^1_{\varepsilon'}$ onto $\mathbb{S}^1$ is the Milnor fibration ϕ in (1). To complete the proof we use the uniform conical structure given in Theorem 1-**iii)** which gives the local triviality of $\mathfrak{F}$ over $\mathbb{C} \setminus \{0\}$. $\square$

3.1. Proof of Theorem 2

Now consider the maps of Theorem 2: $\Psi(x) = (Re(f(x)) : Im(f(x)))$, and $\Phi(x) = \frac{f(x)}{|f(x)|}$. Notice that Φ is a lifting of Ψ to the double cover $\mathbb{S}^1$ of $\mathbb{RP}^1$, so we have the following commutative diagram

$$\begin{tikzcd}
(X \cap \mathbb{B}_\varepsilon) \setminus V \arrow[r, "\Phi"] \arrow[dr, "\Psi"'] & \mathbb{S}^1 \arrow[d] \\
 & \mathbb{RP}^1
\end{tikzcd}$$

From Remark 2.3, for each $\mathcal{L}_\theta \in \mathbb{RP}^1$ and each $\theta \in [0, 2\pi)$ one has,

$$\Psi^{-1}(\mathcal{L}_\theta) = E_\theta \cup E_{\theta+\pi} \quad \text{and} \quad \Phi^{-1}(e^{i\theta}) = E_\theta\ .$$

The vector field w constructed in Proposition 3.4 provides topological trivialisations around the fibres of the maps Ψ and Φ, showing that both are fibre bundles, which proves the first statement of Theorem 2. That the restriction of Φ to the link of X is the classical Milnor fibration is immediate from Theorem 3.

Remark 3.5. If $X \setminus V$ is non-singular the flow obtained in Proposition 3.4 can be made C^∞, thus all the E_θ are diffeomorphic.

On the other hand, let $\mathbb{D}_\eta$ be a disc in $\mathbb{C}$ of radius η where $\varepsilon >> \eta > 0$ and consider the Milnor tube $N(\varepsilon,\eta) = X \cap \mathbb{B}_\varepsilon \cap f^{-1}(\partial\mathbb{D}_\eta)$. Since the restriction of f to $(X \cap \mathbb{B}_\varepsilon) \setminus V$ is a submersion on each stratum, we have that $\{ N(\varepsilon,\eta) \cap S_\alpha \mid S_\alpha \subset X \setminus V \}$ is a Whitney-strong stratification of $N(\varepsilon,\eta)$ [36, Rem. (3.7)]. Since the stratification $\{S_\alpha\}$ of X satisfies Thom's (a_f) condition, if η is small enough, then all the fibres of f in $N(\varepsilon,\eta)$ are transverse to $\mathbb{S}_\varepsilon$. Using Thom-Mather first Isotopy lemma and the transversality of the fibres with the boundary as in [15, §1], we obtain that the restriction of Φ to $N(\varepsilon,\eta)$,

$$\bar{\Phi}\colon N(\varepsilon,\eta) \to \mathbb{S}^1 \,,$$

is also a fibre bundle. This map equals the restriction of f to $N(\varepsilon,\eta)$ followed by the radial projection of $\partial\mathbb{D}_\eta$ onto $\mathbb{S}^1$, so this is the Milnor-Lê fibration (2) up to multiplication by a constant.

It remains to prove that the two fibrations (1) and (2) are equivalent. We need the following, which is a consequence of Lemma 2.7.

Proposition 3.6. *Let $\{S_\alpha\}$ be a Whitney stratification of X adapted to V and for each $\theta \in [0,\pi)$ equip X_θ with the stratification $\{X_\theta \cap S_\alpha\}$ obtained by intersecting X_θ with the strata of $\{S_\alpha\}$. Then for every sufficiently small ball $\mathbb{B}_\varepsilon$ around $\underline{0}$, there exists a stratified, rugose vector field $\tilde{v}$ on $(X \cap \mathbb{B}_\varepsilon) \setminus V$ which has the following properties:*

i) *It is radial, i.e., it is transverse to the intersection of X with all spheres in $\mathbb{B}_\varepsilon$ centred at $\underline{0}$.*
ii) *It is tangent to the strata of each $X_\theta \setminus V$.*
iii) *It is transverse to all the tubes $f^{-1}(\partial\mathbb{D}_\eta)$.*

Proof. Notice that the vector field $v_{\mathfrak{F}}$ in the proof of Lemma 2.8 already satisfies conditions i) and ii) and is rugose. The aim now is to modify this vector field to insure that it satisfies also iii), being rugose.

Let ε_0 be as in Lemma 2.5. Let $\varepsilon_0 > \varepsilon > 0$ be small enough so that the restrictions of f and $\mathfrak{F}$ to $(X \cap \mathbb{B}_\varepsilon) \setminus V$ are both submersions on each stratum; such an ε exists by the theorem of Bertini-Sard-Verdier [36, Thm. (3.3)]. Let u be the radial vector field in the proof of Theorem 3. Now we use f to lift u to a stratified rugose vector field v_f on $(X \cap \mathbb{B}) \setminus V$ such that, for every $x \in (X \cap \mathbb{B}) \setminus V$, we have:

$$df_x(v_f(x)) = u(f(x)) \,.$$

The local flow associated to v_f is transverse to all Milnor tubes $f^{-1}(\partial\mathbb{D}_\eta)$, while the one associated to $v_{\mathfrak{F}}$ is transverse to all spheres in $\mathbb{B}_\varepsilon$ centred at $\underline{0}$. The integral paths of both move along $X_\theta \setminus V$, *i.e.*, along points where the argument θ of f does not change.

From the definitions of v_f and $v_{\mathfrak{F}}$ and the fact that each stratum $S_\alpha \cap X_\theta$ of X_θ is transverse to all the spheres (Lemma 2.5), one can see that the vectors $v_f(x)$ and $v_{\mathfrak{F}}(x)$ cannot point in opposite directions for any $x \in (X \cap \mathbb{B}_\varepsilon) \setminus V$. Hence adding up v_f and $v_{\mathfrak{F}}$ on $(X \cap \mathbb{B}_\varepsilon) \setminus V$ we get a vector field $\tilde{v}$ which satisfies the three properties of Proposition 3.6. □

Now let ϕ denote the restriction of Φ to $L_X \setminus L_f = \mathbb{S}_\varepsilon \cap (X \setminus V)$, which defines the classical Milnor fibration. The flow associated to the vector field $\tilde{v}$ in Proposition 3.6 defines in the usual way a homeomorphism between the fibre of $f^{-1}(e^{i\theta}) \cap \mathbb{B}_\varepsilon$ and the portion of the fibre $\phi^{-1}(e^{i\theta})$ defined by the inequality $|f(x)| \geq \eta$.

To complete the proof we must show that the fibration defined by ϕ on $L_X \setminus L_f$ is equivalent to the restriction of ϕ to the points in the sphere satisfying $|f(x)| > \eta$. For this we use that since f satisfies Thom's (a_f) condition, the restriction of f to $T(\varepsilon, \eta) = \mathbb{S}_\varepsilon \cap f^{-1}(\mathbb{D}_\eta \setminus \{0\})$ is a submersion on each stratum. Hence, again by Verdier's result Proposition 1.11, we can lift the radial vector $u(z) = z$ on $\mathbb{D}_\eta \setminus \{0\}$ to a stratified, rugose vector field on $T(\varepsilon, \eta)$, whose flow preserves the fibres of ϕ and is transverse to the intersection with $\mathbb{S}_\varepsilon$ of all the Milnor tubes $f^{-1}(\partial \mathbb{D}_{\eta'})$ for all $0 < \eta' \leq \eta$. This gives the equivalence of the two fibrations, and therefore finishes the proof of Theorem 2.

4. A fibration theorem on the blow up

In this section we complete the proof of statement **i)** of Theorem 1 by proving that the varieties X_θ are all homeomorphic. In order to do that, we realise the spaces X_θ as fibres of a topological fibre bundle.

A minimal way to obtain an unfolding of the pencil (X_θ) is achieved by the blow-up of its axis $V = f^{-1}(0)$. We produce in this way a new analytic space $\tilde{X}$ with a projection to $\mathbb{RP}^1$ whose fibres are exactly the X_θ's. The space $\tilde{X}$ is equipped with a Whitney stratification obtained in a canonical way from the one on X. Using the Thom-Mather First Isotopy Lemma, we prove that $\tilde{X}$ is a fibre bundle over $\mathbb{RP}^1$. This is a new Milnor-type fibration theorem in which we do not need any more to remove the zero locus of the function f.

4.1. The real blow up

As before, we consider a sufficiently small representative X of the complex analytic germ $(X, \underline{0})$ and $f \colon (X, \underline{0}) \to (\mathbb{C}, 0)$ holomorphic. Set $V = f^{-1}(0)$ and consider the real analytic map

$$\begin{array}{rccl} \Psi \colon & X \setminus V & \to & \mathbb{RP}^1 \\ & z & \mapsto & (Re(f(z)) : Im(f(z)))\,. \end{array}$$

Let $\tilde{X}$ be the analytic set in $X \times \mathbb{RP}^1$ defined by $Re(f)t_2 - Im(f)t_1 = 0$, where $(t_1 : t_2)$ is a system of homogeneous coordinates in $\mathbb{RP}^1$. The first projection induces a real analytic map:

$$e_V : \tilde{X} \to X\,;$$

this is the real blow-up of V in X [20, §3]. It induces a real analytic isomorphism $\tilde{X} \setminus e_V^{-1}(V) \cong X \setminus V$. The inverse image of V by e_V is $V \times \mathbb{RP}^1$.

The second projection induces a real analytic map:

$$\tilde{\Psi} : \tilde{X} \to \mathbb{RP}^1\,,$$

and one has: $\tilde{\Psi} \mid_{\tilde{X}\setminus e_V^{-1}(V)} = \Psi \circ e_V \mid_{\tilde{X}\setminus e_V^{-1}(V)}$, *i.e.*, $\tilde{\Psi}$ extends Ψ to $e_V^{-1}(V) \cong V \times \mathbb{RP}^1$. It is clear that each fibre $\tilde{\Psi}^{-1}(t)$ is isomorphic to $X_\theta \times \{t\}$, where θ is the angle between the horizontal axis and the line represented by t.

4.2. The fibration theorem on the blow up

We now prove Theorem 5, *i.e.*, that

$$\tilde{\Psi} : \tilde{X} \to \mathbb{RP}^1\,,$$

is a topological fibre bundle with fibres the X_θ.

For this, we consider a Whitney stratification $\{S_\alpha\}$ of X adapted to V. This induces a stratification Σ_α on $\tilde{X}$ defined by:

$$\Sigma_\alpha = e_V^{-1}(S_\alpha).$$

We claim this stratification is Whitney regular. This is Proposition 4.3 at the end of the section, and we assume it for the moment.

Notice that the map $\tilde{\Psi}$ restricted to each stratum Σ_α has no critical point. In fact, outside $e_V^{-1}(V)$, $\tilde{\Psi}$ coincides with Ψ (up to isomorphism), which is a submersion onto $\mathbb{RP}^1$ restricted to each stratum of $X \setminus V$. On the other hand, the restriction of $\tilde{\Psi}$ to a stratum contained in $e_V^{-1}(V)$ is surjective onto $\mathbb{RP}^1$ and is given by the second projection on the product $V \times \mathbb{RP}^1$.

In order to be able to apply Thom-Mather's First Isotopy Lemma we need to restrict the map $\tilde{\Psi}$ to a compact subset where it is a proper submersion (see [20, Proposition 11.1]).

Consider the closed ball $\bar{\mathbb{B}}_\varepsilon$ centred at $\underline{0}$ with radius ε, in the ambient space $\mathbb{C}^n$. The intersection $\tilde{X}$ with $\bar{\mathbb{B}}_\varepsilon \times \mathbb{RP}^1$ is compact and hence the restricted map $\tilde{\Psi} : \tilde{X} \cap (\bar{\mathbb{B}}_\varepsilon \times \mathbb{RP}^1) \to \mathbb{RP}^1$ is proper. This map is a submersion in the stratified sense if and only if the fibres of $\tilde{\Psi}$ are transverse to $\mathbb{S}_\varepsilon \times \mathbb{RP}^1$ or, equivalently, the spaces X_θ are all transverse to $\mathbb{S}_\varepsilon$ for sufficiently small ε. But this is already given by Lemma 2.5.

Theorem 5 obviously implies that the spaces X_θ are all homeomorphic.

Remark 4.1. The usual proof of the Thom-Mather First Isotopy Lemma uses that the map $\tilde{\Psi}$ is a proper submersion on each stratum, in order to lift vector fields from the target space, in this case $\mathbb{RP}^1$. In the case when both X and f have an isolated singularity at $\underline{0}$, then each X_θ also has an isolated singularity at $\underline{0}$ and the canonical vector field on $\mathbb{R}P^1 \cong \mathbb{S}^1$ can be lifted to a rugose vector field $\widetilde{v}$ on the blow-up $\tilde{X}$, which is C^∞ away from the stratum $\{0\} \times \mathbb{R}P^1$ and leaves invariant the subspace $V \times \mathbb{R}P^1$. This will be used later, for proving Corollary 4.

Remark 4.2. Theorem 5 gives another proof that the stratification induced on the X_θ is Whitney regular. Since Whitney regularity conditions are stable under transversality (see for example [9, Chap. 1 (1.4)]), the fibres of the projection $\tilde{\Psi} : \tilde{X} \to \mathbb{RP}^1$ inherit a Whitney regular stratification given by $\{\Sigma_\alpha \cap \tilde{\Psi}^{-1}(t)\}$. Hence $\{S_\alpha \cap X_\theta\}$ is a Whitney regular stratification of each X_θ.

4.3. An induced Whitney stratification on the blow up

The proposition below completes the proofs of Theorems 1–**i)** and 5.

Proposition 4.3. *Consider a Whitney stratification $\{S_\alpha\}$ of a sufficiently small representative X of the germ $(X, \underline{0})$ adapted to V. Then the partition $\{\Sigma_\alpha\}$ given by the inverse images by e_V of the strata S_α is a Whitney stratification of $\tilde{X}$.*

Proof. Note that if S_α is a stratum contained in V, the corresponding stratum $\Sigma_\alpha \subset \tilde{X}$ is $S_\alpha \times \mathbb{RP}^1$.

We first check that this partition of $\tilde{X}$ satisfies the frontier condition. Let S_α and S_β be strata contained respectively in V and $\operatorname{Sing}(X)$, such that $\Sigma_\alpha \cap \overline{\Sigma_\beta} \neq \emptyset$. The corresponding strata in X satisfy the inclusion $S_\alpha \subset \overline{S_\beta}$.

Let $(y, t_1 : t_2)$ be a point of Σ_α, *i.e.* $f(y) = 0$ and $y \in \overline{S_\beta}$. Since S_β is not contained in V, the restriction of f to a neighbourhood of y in $\overline{S_\beta}$ is surjective over a neighbourhood of the origin in $\mathbb{C}$. Hence for every line $\mathcal{L}_\theta \subset \mathbb{R}^2$ containing the origin, the intersection $X_\theta \cap S_\beta$ has points y_n arbitrarily close to y. Taking the line corresponding to the projective point $(t_1 : t_2)$, we obtain a sequence $(y_n, t_1 : t_2) \in \Sigma_\beta$ converging to $(y, t_1 : t_2)$.

In the other cases, it is clear that the frontier condition holds.

Now, we prove that this stratification is Whitney regular. Consider a stratum $\Sigma_\alpha = S_\alpha \times \mathbb{RP}^1 \subset V \times \mathbb{RP}^1$ and a stratum Σ_β not contained in $V \times \mathbb{RP}^1$ such that $\Sigma_\alpha \subset \bar{\Sigma}_\beta$. For the other cases it is easy to see that Whitney conditions hold.

Consider a sequence of points $(x_n, t_n) \in \Sigma_\beta \subset X \times \mathbb{RP}^1$ converging to a point $(x, t) \in S_\alpha \times \mathbb{RP}^1$ such that the sequence of tangent spaces $T_n := T_{(x_n, t_n)}\Sigma_\beta$ converges to a linear space T.

Lemma 4.4. *For any $b \in \mathbb{R}$ there exists a vector $a \in \mathbb{C}^n$ such that the vector $(a, b) \in T$.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
& \tilde{X} \arrow[r, "e_V"] \arrow[d, "\pi"'] & X \arrow[d, "\psi"] \\
U \times \mathbb{RP}^1 \supset & \tilde{U} \arrow[r, "e_0"] & U \subset \mathbb{R}^2
\end{tikzcd}$$

where ψ is the real analytic map on X defined by $(Re(f), Im(f))$, $e_0 : \tilde{U} \to U$ is the blow-up of the origin in a neighbourhood U of 0 in $\mathbb{R}^2$ and π is the pull-back of ψ by e_0.

Let us work on a local chart $X \times \mathbb{R}$ of $X \times \mathbb{RP}^1$ near the point (x, t). Call $(y_n, t_n) \in \tilde{U}$ and $(0, t) \in \{0\} \times \mathbb{R}$ the respective images of (x_n, t_n) and (x, t) by π. The direction of the tangent space $T_{(0,t)}(\{0\} \times \mathbb{R}) = \{0\} \times \mathbb{R}$ is naturally contained in the limit of directions of tangent spaces $\mathcal{T}_n := T_{(y_n, t_n)}\tilde{U}$. Hence, there exists a sequence of directions of lines $l_n \subset \mathcal{T}_n$ converging to $\{0\} \times \mathbb{R}$.

Since the map π is a submersion at (x_n, t_n), there exists a sequence of directions of lines $h_n \subset T_n$ whose images by the tangent map to π at (x_n, t_n) is l_n.

Since l_n converges to $\{0\} \times \mathbb{R}$, the limit h of h_n is not contained in $\mathbb{C}^n \times \{0\}$. So there exists a vector $(a, b) \in T$ such that that $b \neq 0$.

$\square$

Lemma 4.5. *The stratification $\{\Sigma_\alpha\}$ satisfies Whitney's (a)-condition.*

Proof. Under the previous notations, we need to prove that the direction of tangent space $T_{(x,t)}\Sigma_\alpha$ is contained in T.

Let us now describe the tangent space T_n. Recall that the blow-up $\tilde{X}$ is defined as the subspace of $X \times \mathbb{RP}^1$ given by the equation $t_2 Re(f) - t_1 Im(f) = 0$, where $(t_1 : t_2)$ are homogeneous coordinates in $\mathbb{RP}^1$.

Suppose the point $(x, t) \in S_\alpha \times \mathbb{RP}^1$ lives in the local chart given by $t_1 \neq 0$ or, equivalently, $Re(f) \neq 0$, and consider a holomorphic extension $\tilde{f}$ of f to the ambient space $\mathbb{C}^n$. Denote by f_1 and f_2 respectively the real and imaginary parts of $\tilde{f}$.

We then have:

$$T_n = \{(a, b) \in T_{x_n} S_\beta \times \mathbb{R}, \langle (\frac{t_2}{t_1} \operatorname{grad}_{x_n} f_1 - \operatorname{grad}_{x_n} f_2)\,,\, a \rangle + f_1(x_n) b = 0\}$$

where $\langle\ ,\ \rangle$ is the real scalar (or inner) product in a real vector space.

If we call u_n the vector $((\frac{t_2}{t_1} \operatorname{grad}_{x_n} f_1 - \operatorname{grad}_{x_n} f_2), f_1(x_n)) \in \mathbb{C}^n \times \mathbb{R}$, L_n the line generated by u_n and $N(L_n)$ its orthogonal linear space in $\mathbb{C}^n \times \mathbb{R}$, then we can write:

$$T_n = (T_{x_n} S_\beta \times \mathbb{R}) \cap N(L_n).$$

We now prove that the sequence of lines L_n converges to a line L contained in $\mathbb{C}^n \times \{0\}$. This is a consequence of the Łojasiewicz inequality [17, p. 92].

In fact, if g is a real analytic map in a neighbourhood of a point $p \in \mathbb{R}^n$ then there exist a neighbourhood $p \in W \subset \mathbb{R}^n$ and $0 < \theta < 1$ such that, for any $q \in W$ we have:

$$\|f(q) - f(p)\|^\theta \leq \|\operatorname{grad}_q f\|. \tag{21}$$

We will apply this inequality to f_1 in a neighbourhood of x.

Since $\tilde{f}$ is holomorphic, the vectors $\operatorname{grad} f_1$ and $\operatorname{grad} f_2$ are orthogonal and have the same module at any point, so:

$$\|\frac{t_2}{t_1} \operatorname{grad}_{x_n} f_1 - \operatorname{grad}_{x_n} f_2\|^2 = \|\operatorname{grad}_{x_n} f_1\|^2 ((\frac{t_2}{t_1})^2 + 1).$$

The function $\tilde{f}$ being holomorphic, it has an isolated critical value at 0 and hence $\operatorname{grad}_{x_n} f_1 \neq 0$. So dividing the vector u_n by the module of $\operatorname{grad}_{x_n} f_1$ we have that

$$\frac{\|\frac{t_2}{t_1} \operatorname{grad}_{x_n} f_1 - \operatorname{grad}_{x_n} f_2\|}{\|\operatorname{grad}_{x_n} f_1\|}$$

tends to a non-zero value, while by the inequality (21) one has:

$$\frac{\|f_1(x_n)\|}{\|\operatorname{grad}_{x_n} f_1\|} < \|f_1(x_n)\|^{1-\theta}\,,$$

and hence it tends to zero. So the sequence of lines L_n tends to a line L contained in $\mathbb{C}^n \times \{0\}$ and the normal spaces $N(L_n)$ converge to a linear space containing $\{0\} \times \mathbb{R}$.

We are now going to prove that the limit T of T_n is equal to the intersection of the limit of $T_{x_n} S_\beta \times \mathbb{R}$ with the normal space $N(L)$ to L in $\mathbb{C}^n \times \mathbb{R}$.

Since the linear space $N(L)$ is a hyperplane it is sufficient to show that the limit of $T_{x_n} S_\beta \times \mathbb{R}$ is not contained in $N(L)$, and then the limit of the intersection is the intersection of the limits.

By Lemma 4.4, for any $0 \neq b \in \mathbb{R}$ there exists a vector $a \in \mathbb{C}^n$ such that $(a, b) \in T$. Since S_β has real dimension at least two, then there exists $b \in \mathbb{R}$, $\neq 0$ and there exists $a \in \mathbb{C}^n$, $\neq 0$, such that $(a, b) \in T$.

So there exists a sequence $(a_n, b_n) \in T_n$ converging to (a, b). This means that $a_n \in T_{x_n} S_\beta$ and the scalar product $\langle (a_n, b_n)\, ,\, u_n \rangle = 0$. If we write $u_n = (u_{n,1}, u_{n,2}) \in \mathbb{C}^n \times \mathbb{R}$, then

$$b_n = -\langle a_n \, , \, \frac{u_{n,1}}{u_{n,2}} \rangle.$$

Since b_n converges to a non zero real value, the limit a of a_n is not orthogonal in $\mathbb{C}^n$ to the limit D of lines generated by the vectors $\frac{u_{n,1}}{u_{n,2}}$. Notice that we have the equality $L = D \times \{0\}$. So for any $s \in \mathbb{R}$ the vector (a, s) is not orthogonal to the line L, and then

$$\lim T_{x_n} S_\beta \times \mathbb{R} \not\subseteq N(L).$$

We conclude that

$$T = ((\lim T_{x_n} S_\beta) \times \mathbb{R}) \cap N(L).$$

On the other hand, the direction of tangent space to Σ_α at (x, t) is given by:

$$T_{(x,t)} \Sigma_\alpha = T_x S_\alpha \times T_t \mathbb{RP}^1.$$

Since the stratification $\{S_\alpha\}$ on X satisfies Whitney's (a)-condition we have

$$T_x S_\alpha \subset \lim T_{x_n} S_\beta \, ,$$

and hence we obtain

$$T_{(x,t)} \Sigma_\alpha \subset \lim T_{(x_n, t_n)} \Sigma_\beta \, ,$$

which proves the (a)-condition. □

We now finish the proof of Proposition 4.3. We use condition (a) to prove condition (b).

Keeping the previous notations, consider a sequence of points $(y_n, s_n) \in \Sigma_\alpha = S_\alpha \times \mathbb{RP}^1$ converging to $(x, t) \in \Sigma_\alpha$ such that the sequence of lines l_n joining the points (x_n, t_n) and (y_n, s_n) in $\mathbb{C}^n \times \mathbb{R}$ converges to a line l. We need to prove that $l \subset T$.

Consider the line h_n generated in $\mathbb{C}^n$ by the vector $x_n - y_n$. We can suppose the sequence of lines h_n converges to a line h. The lines h_n and h are respectively

the projection onto $\mathbb{C}^n$ of the lines l_n and l. So if $(u, v) \in \mathbb{C}^n \times \mathbb{R}$ is a directing vector of the line l, then u is a directing vector of h.

Since the strata S_β and S_α satisfy Whitney's (b)-condition, the line h is contained in $\lim T_{x_n} S_\beta$, and hence, there exists $v' \in \mathbb{R}$ such that the vector $(u, v') \in \mathbb{C}^n \times \mathbb{R}$ is actually in the limit T of tangent spaces T_n.

By Lemma 4.5, the real line $\{0\} \times \mathbb{R}$ is contained in T. So the vector $(u, v') + (0, v - v') \in T$, and we obtain $l \subset T$, completing the proof of Proposition 4.3. $\square$

4.4. Proof of Corollary 4

Recall from Remark 2.3 that we can write

$$X_\theta = E_\theta \cup V \cup E_{\theta+\pi} \ .$$

Lemma 4.6. *Assume $(X, \underline{0})$ is irreducible and for every angle θ let $\overline{E}_\theta$ denote the topological closure of E_θ. Then we have:*

$$\overline{E}_\theta = E_\theta \cup V.$$

Proof. Since f is not constant in $(X, \underline{0})$ and this germ is irreducible, the subspace $V = f^{-1}(0)$ has complex codimension 1 in $(X, \underline{0})$. So for each point $x \in V$ there exists a neighbourhood $U \subset \mathbb{C}^n$, such that the restriction $f : U \cap X \to \mathbb{C}$ is surjective onto an open neighbourhood of 0 in $\mathbb{C}$. Hence, for every line $\mathcal{L}$ through $\underline{0}$, each half line in $\mathcal{L} \setminus \{0\}$ intersects the image of the restriction of f in an open segment. In other words, for each θ, E_θ has a non-empty intersection with $X \cap U$. Choosing the neighbourhood U arbitrarily small, we get that for each θ, there exists a sequence of points in E_θ converging to x. That is, every $x \in V$ is in $\overline{E}_\theta$. Since $E_\theta \cup V$ is closed, we obtain the equality of the lemma. $\square$

We see from the preceding lemma that E_θ and $E_{\theta+\pi}$ have a common border on V. It is in this sense that we say that they are glued together along V forming the subspace X_θ.

Notice that the fibres of ϕ are precisely the intersections of the sphere $\mathbb{S}_\varepsilon$ with the corresponding E_θ. So the fibres of ϕ over two antipodal points of $\mathbb{S}^1$ are glued together along the link L_f forming the link $K_\theta = X_\theta \cap \mathbb{S}_\varepsilon$ of X_θ. The fact that the link K_θ is homeomorphic to the link of $\{Re\, f = 0\}$ is an immediate consequence of Theorem 5. Furthermore, if both X and f have an isolated singularity at $\underline{0}$, then by Remark 4.1, all $X_\theta \setminus \{\underline{0}\}$ are actually diffeomorphic.

It remains to prove the last statement in Corollary 4, *i.e.*, that if both X and f have an isolated singularity at $\underline{0}$, then the link K_θ of each X_θ is diffeomorphic to the double of the Milnor fibre of f. This follows from the previous discussion and the *folklore* theorem, that in this setting, Milnor's fibration theorem gives an open-book decomposition of the sphere. In fact, from the discussion above (see equation (5) in Remark 2.3) we have:

$$K_\theta = (E_\theta \cap \mathbb{S}_\varepsilon) \cup (V \cap \mathbb{S}_\varepsilon) \cup (E_{\theta+\pi} \cap \mathbb{S}_\varepsilon) \ .$$

Then, by Remark 4.1, we have that each K_θ is a smooth real analytic, oriented manifold of dimension $2\dim_{\mathbb{C}} X - 2$, obtained by gluing two Milnor fibres of

f along their boundary by a smooth diffeomorphism, given by a smooth flow. Such a diffeomorphism is necessarily isotopic to the identity, and therefore each K_θ is diffeomorphic to the double of the Milnor fibre of f.

Remark 4.7. The hypothesis in Corollary 4 of X being irreducible avoids situations as in the following example. Let $g : \mathbb{C}^2 \to \mathbb{C}$ be defined by $g(x, y) = x \cdot y$. Its zero locus X consists of the two axis. Now let $f : X \to \mathbb{C}$ be given by $f(x, y) = x$. Then $V \subset X$ is the y-axe and each X_θ is

$$X_\theta = \{(x, y) \in \mathbb{C}^2 \,|\, x = 0 \text{ or } y = 0 \text{ and } x = te^{i\theta}\,, t \in \mathbb{R}\,\}.$$

One has $E_\theta \cup E_{\theta+\pi} = \{(x, y) \in \mathbb{C}^2 \,|\, y = 0 \text{ and } x = te^{i\theta}\,, t \in \mathbb{R}\,\}$. Thus no $z \in V \setminus \{\underline{0}\}$ is in the closure of $E_\theta \cup E_{\theta+\pi}$.

Remark 4.8. If X is non-singular and f has an isolated critical point with respect to some Whitney stratification, then, as observed in [26], the family $\{X_\theta\}$ is (c)-regular in Bekka's sense (see [1]). In this case, one can follow the method of [26] to construct two flows on the product $X_\theta \setminus \{0\} \times [0, \pi)$ which give on the one hand the transversality of the X_θ's with the spheres, and on the other hand a flow interchanging the $X_\theta \setminus \{0\}$'s, as in Proposition 3.4. This is similar to what we do here on the blow-up, and in fact these considerations somehow inspired our constructions.

5. The real analytic case

We now look at real analytic mappings from the viewpoint of the previous sections. Let U be an open neighbourhood of the origin $\underline{0}$ in $\mathbb{R}^{n+2}$ and let $f : (U, \underline{0}) \to (\mathbb{R}^2, 0)$ be a locally surjective, real analytic map. Set as before $V := f^{-1}(0)$ and denote by $K_\varepsilon = K$ the intersection $V \cap \mathbb{S}_\varepsilon$. This is the link of V, which is independent of ε up to homeomorphism. We assume further that $0 \in \mathbb{R}^2$ is the only critical value of f, so the Jacobian matrix $Df(x)$ has rank 2 for all $x \in U \setminus V$.

5.1. The strong Milnor condition

We know from [21, 22] that if f has an isolated critical point at $\underline{0}$, then one has a fibration of the Milnor-Lê type (2), and this can always be taken into a fibration of the complement of K, $\mathbb{S}_\varepsilon \setminus K \xrightarrow{\phi} \mathbb{S}^1$. So K is a *fibred knot*. But we also know from [22] that the projection map ϕ can not be always taken to be the obvious map $f/|f|$. As noticed in [25], these remarks extend to the case when the real analytic map-germ f is assumed to have only an isolated critical value at $0 \in \mathbb{R}^2$ provided V has dimension greater than 0 and f has the Thom property, *i.e.*, when there exists a Whitney stratification of U adapted to V, for which f satisfies Thom's a_f-condition.

The following definitions extend those given in [27] when f has an isolated critical point.

Definition 5.1. Let $f : (U, \underline{0}) \to (\mathbb{R}^2, 0)$ be a locally surjective real analytic map-germ.

i) We say that f has the Milnor-Lê property at $\underline{0} \in U \subset \mathbb{R}^{n+2}$ if it has an isolated critical value at $0 \in \mathbb{R}^2$, V has dimension more than 0 and f has the Thom property.
ii) We say that f has the strong Milnor property if for every sufficiently small $\varepsilon > 0$ one has a C^∞ fibre bundle $\mathbb{S}_\varepsilon \setminus K_\varepsilon \xrightarrow{\phi} \mathbb{S}^1$, where the projection map ϕ is $f/|f|$. (If one considers map-germs defined on analytic varieties with singular set of dimension greater than 0, then this fibre bundle is required to be only continuous.)

5.2. d-Regularity for real analytic map-germs

Following the construction above of a canonical pencil for holomorphic maps, for each line $\mathcal{L}_\theta$ through $0 \in \mathbb{R}^2$, let $X_\theta = f^{-1}(\mathcal{L}_\theta)$. One has:

Proposition 5.2. *Each X_θ is a real analytic hypersurface of U, of codimension* 1*, such that:*

- *Their union is U and the intersection of any two distinct X_θ's is V.*
- *Each X_θ is non-singular away from the singular set of V,* $\mathrm{Sing}(V)$*.*

The proof is an exercise and we leave it to the reader.

Definition 5.3. The family $\{X_\theta | \mathcal{L} \in \mathbb{RP}^1\}$ is *the canonical pencil* of f.

Definition 5.4. The map f is *d-regular at* $\underline{0}$ if there exist a positive definite metric $d : \mathbb{R}^{n+2} \to \mathbb{R}$, defined by some quadratic form, and $\varepsilon > 0$ such that every sphere (for the metric d) of radius $\leq \varepsilon$ and centred at $\underline{0}$ meets transversally each X_θ. If we want to emphasise the metric, then we say that f is *d-regular at* $\underline{0}$ *with respect to the given metric.*

Examples 5.5. i) By [22] (see Lemma 2.5 above), every holomorphic germ $f : (\mathbb{C}^n, \underline{0}) \to (\mathbb{C}, 0)$ is d-regular at $\underline{0}$ for the usual metric.

ii) By [25], given holomorphic germs $f, g : (\mathbb{C}^2, \underline{0}) \to (\mathbb{C}, 0)$ such that $f\bar{g}$ has an isolated critical value at $0 \in \mathbb{C}$, the map $f\bar{g}$ is d-regular at $\underline{0}$ for the usual metric. The same statement holds for the map $f/g : (\mathbb{C}^n, \underline{0}) \to (\mathbb{C}, 0)$ if we further demand that the meromorphic germ f/g be semi-tame (see [25] for the definition and details).
iii) By [31], every twisted Pham-Brieskorn polynomial $z_1^{a_1}\, \bar{z}_{\sigma(1)} + \cdots + z_n^{a_n}\, \bar{z}_{\sigma(n)}$, where σ is a permutation of $\{1, ..., n\}$, is d-regular at $\underline{0}$ for the usual metric. The same statement holds for all quasi-homogeneous singularities, since the $\mathbb{R}^+$-orbits are tangent to the X_θ. This applies, for instance, to the singularities with polar action of [5, 23].
iv) By [26], every map-germ $g \colon (\mathbb{R}^{n+2}, \underline{0}) \to (\mathbb{R}^2, 0)$ for which its pencil is c-regular (in the sense of K. Bekka) with respect to the control function defined by the metric d, is d-regular. That was indeed one of the motivations to pursue this research in the real analytic setting.

5.3. The fibration theorem

Let us prove Theorem 6, stated in the introduction. Notice that statement i) is well-known (see for instance [25]), so we only need to prove statements ii) and iii).

Let us equip U with a Whitney stratification adapted to V, and consider the restriction of f to $\mathbb{B}_\varepsilon \setminus V$, which is a submersion by hypothesis.

Define the real analytic map

$$\mathfrak{F}\colon \mathbb{B}_\varepsilon \setminus V \to \mathbb{R}^2 \setminus \{0\}$$

by

$$\mathfrak{F}(x) = \|x\| \frac{f(x)}{\|f(x)\|}.$$

Notice that given $y = \mathfrak{F}(x)$ in a line $\mathcal{L}_\theta$ through 0, the fibre $\mathfrak{F}^{-1}(y)$ is the intersection of the corresponding element X_θ in the pencil with the sphere of radius $\|x\|$ centred at $\underline{0}$. Thus we call $\mathfrak{F}$ the *spherefication* of f, as in the holomorphic case.

Lemma 5.6. *If f is d-regular, then $\mathfrak{F}$ is a submersion for all $x \in B_\varepsilon \setminus V$ with $\varepsilon > 0$ sufficiently small.*

The proof of this lemma is exactly the same as in Lemma 2.7

As in the holomorphic case, this lemma implies the proposition below; we leave the details to the reader. The only point to notice is that because we are assuming the ambient space U is smooth, the liftings we use of vector fields from $\mathbb{R}^2 \setminus 0$ to U can be taken to be C^∞ and we do not need to use Verdier's theory of rugose vector fields.

Proposition 5.7. *If f is d-regular for some metric d, then there exists $\varepsilon > 0$ sufficiently small, such that there exists C^∞ vector field on $\mathbb{B}_\varepsilon \setminus V$ such that :*

i) *Each of its integral lines is contained in an element X_θ of the pencil;*
ii) *It is transverse to all d-spheres around $\underline{0}$; and*
iii) *It is transverse to all Milnor tubes $f^{-1}(\partial \mathbb{D}_\delta)$, for all sufficiently small discs $\mathbb{D}_\delta$ centred at $0 \in \mathbb{R}^2$.*

Remark 5.8 (Uniform Conical Structure)**.** Notice that d-regularity for f implies that its canonical pencil has a uniform conical structure away from V. In the holomorphic case one has uniform conical structure everywhere near $\underline{0}$, this is part of the content of Theorem 1. In the real analytic case envisaged here, one can prove this uniform conical structure everywhere near $\underline{0}$ if we further demand that f has the strict Thom property, which is automatic for holomorphic maps into $\mathbb{C}$.

Let us now prove Theorem 6. Recall that we are assuming f has the Thom a_f property. Thus for $\varepsilon >> \delta > 0$ sufficiently small one has a *solid* Milnor tube,

$$SN(\varepsilon, \delta) := \mathbb{B}_\varepsilon \cap f^{-1}(\mathbb{D}_\delta \setminus \{0\})\,,$$

and a fibre bundle:

$$f : SN(\varepsilon, \delta) \longrightarrow \mathbb{D}_\delta \setminus \{0\}\,,$$

where $\mathbb{B}_\varepsilon$ is the ball in $\mathbb{R}^{n+2}$ of radius ε and centre $\underline{0}$ and $\mathbb{D}_\delta$ is the disc in $\mathbb{R}^2$ of radius δ and centre 0. The restriction of this locally trivial fibration to the boundary of $\mathbb{D}_\delta$ gives the fibration in the first statement in Theorem 6.

Now let $\pi_1 : \mathbb{D}_\delta \setminus \{0\} \to \mathbb{S}^1$ be defined by $t \mapsto t/|t|$. Let $\pi_2 : \mathbb{S}^1 \to \mathbb{R}P^1$ be the canonical projection, and set

$$\Psi := \pi_2 \circ \pi_1 \circ f : SN(\varepsilon, \delta) \longrightarrow \mathbb{R}P^1\,.$$

This is a fibre bundle with fibres $X_\theta \cap SN(\varepsilon, \delta)$, and it yields to statement (ii) in Theorem 6 restricted to the solid Milnor tube $SN(\varepsilon, \delta)$. We now use the vector field in Proposition 5.7 to complete the proof of Theorem 6.

APPENDIX: Proof of Lemma 3.2

Here we use the notations and hypotheses of the lemma in question.

PROOF OF LEMMA 3.2: Suppose there were points $z \in S_\alpha$ arbitrarily close to the origin with

$$\pi_{\alpha_z}\big(\operatorname{grad}\log f(z)\big) = \lambda \pi_{\alpha_z}(z) \neq 0,$$

and with $|\arg \lambda|$ strictly greater than $\pi/4$. In other words, λ lies in the open half-plane

$$\Re\big((1+i)\lambda\big) < 0,$$

or the open half-plane

$$\Re\big((1-i)\lambda\big) < 0.$$

We want to express these conditions by *real analytic* equalities and inequalities, so as to apply the *analytic curve selection lemma* [4].

Let

$$W = \{\, z \in S_\alpha \mid \pi_{\alpha_z}\big(\operatorname{grad}\log f(z)\big) = \mu \pi_{\alpha_z}(z),\ \mu \in \mathbb{C}\,\}.$$

Thus $z \in W$ if and only if the equations

$$\pi_{\alpha_z}(z)_j\Big(\pi_{\alpha_z}\big(\operatorname{grad} f(z)\big)_k\Big) = \pi_{\alpha_z}(z)_k\Big(\pi_{\alpha_z}\big(\operatorname{grad} f(z)\big)_j\Big),$$

are satisfied, where the subindices j and k denote the j-th and k-th components.

Setting $z_j = x_j + y_j$ and taking real and imaginary parts, we obtain a collection of real *analytic* equations in the real variables x_j and y_j. This proves

that $W \subset S_\alpha \subset \mathbb{C}^n$ is a *real analytic set*. Note that a point $z \in S_\alpha$ belongs to W if and only if

$$\pi_{\alpha_z}\big(\mathrm{grad}\, f(z)\big)/\bar{f}(z) = \lambda \pi_{\alpha_z}(z)$$

for some complex number λ. Multiplying by $\bar{f}(z)$ and taking the inner product with $\bar{f}(z)\pi_{\alpha_z}(z)$, we have

$$\begin{aligned}\Big\langle \pi_{\alpha_z}\big(\mathrm{grad}\, f(z)\big), \bar{f}(z)\pi_{\alpha_z}(z)\Big\rangle &= \Big\langle \lambda \bar{f}(z)\pi_{\alpha_z}(z), \bar{f}(z)\pi_{\alpha_z}(z)\Big\rangle \\ &= \lambda \|\bar{f}(z)\pi_{\alpha_z}(z)\|^2.\end{aligned}$$

In other words, the number λ, multiplied by a positive real number, is equal to

$$\lambda'(z) = \Big\langle \pi_{\alpha_z}\big(\mathrm{grad}\, f(z)\big), \bar{f}(z)\pi_{\alpha_z}(z)\Big\rangle.$$

Hence

$$\arg \lambda = \arg \lambda'.$$

Clearly λ' is a (complex valued) real *analytic* function of the real variables x_j and y_j. Now let U_+ (respectively U_-) denote the open set consisting of all z satisfying the real *analytic* inequality

$$\Re\big((1+i)\lambda'(z)\big) < 0$$

(respectively

$$\Re\big((1-i)\lambda'(z)\big) < 0$$

for U_-).

We have assumed that there exist points z arbitrarily close to the origin with $z \in W \cap (U_+ \cup U_-)$. Here by the *analytic curve selection lemma* (see for instance [4, Proposition 2.2]), there must exist a real analytic path

$$p\colon [0, \varepsilon) \to S_\alpha \subset \mathbb{C}^n$$

with $p(0) = \underline{0}$ and with either

$$p(t) \in W \cap U_+$$

for all $t > 0$, or

$$p(t) \in W \cap U_-$$

for all $t > 0$. In either case, for each $t > 0$ we get

$$\pi_{\alpha_t}\Big(\mathrm{grad} \log f\big(p(t)\big)\Big) = \lambda \pi_{\alpha_t}\big(p(t)\big)$$

with

$$|\arg \lambda(t)| > \pi/4$$

which contradicts Lemma 3.1. $\square$

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